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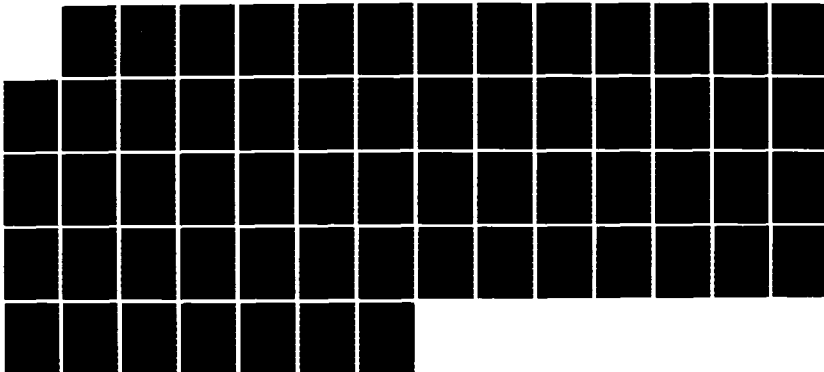
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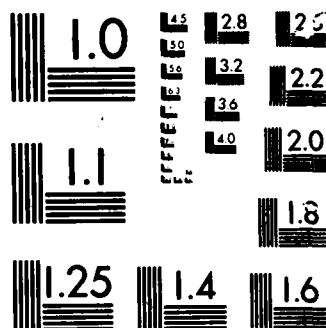
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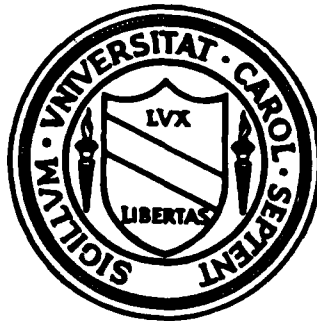
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STOCHASTIC INTEGRATION FOR OPERATOR VALUED
PROCESSES ON HILBERT SPACES AND ON NUCLEAR SPACES

by
H. Korezlioglu
and
C. Martias

Technical Report No. 85

March 1986 revision

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SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED			1b. RESTRICTIVE MARKINGS		
2a. SECURITY CLASSIFICATION AUTHORITY NA			3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for Public Release; Distribution Unlimited		
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE NA					
4. PERFORMING ORGANIZATION REPORT NUMBER(S) Technical Report No. 85			5. MONITORING ORGANIZATION REPORT NUMBER(S) AFOSR-TR- 86 - 0326		
6a. NAME OF PERFORMING ORGANIZATION University of North Carolina		6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION AFOSR/NM		
6c. ADDRESS (City, State and ZIP Code) Center for Stochastic Processes, Statistics Department, Phillips Hall 039-A, Chapel Hill, NC 27514			7b. ADDRESS (City, State and ZIP Code) Bldg. 410 Bolling AFB, DC 20332-6448		
8a. NAME OF FUNDING/SPONSORING ORGANIZATION AFOSR		8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER F49620 85 C 0144		
8c. ADDRESS (City, State and ZIP Code) Bldg. 410 Bolling AFB, DC			10. SOURCE OF FUNDING NOS.		
			PROGRAM ELEMENT NO. 6.1102F	PROJECT NO. 2304	TASK NO. A5
11. TITLE (Include Security Classification) Stochastic Integration for Operator Valued Processes on Hilbert Spaces and on Nuclear Spaces					
12. PERSONAL AUTHOR(S) H. Korezlioglu and C. Martias					
13a. TYPE OF REPORT technical		13b. TIME COVERED FROM <u>9/85</u> TO <u>8/86</u>		14. DATE OF REPORT (Yr., Mo., Day) March 1986 revision	
				15. PAGE COUNT 57	
16. SUPPLEMENTARY NOTATION <i>Supersedes AD A158 878</i>					
17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)		
FIELD	GROUP	SUB. GR.	Keywords: Stochastic integration, Hilbert space valued square integrable martingales, representation of square integrable martingales.		
XXXXXXXXXXXXXXXXXXXX					
19. ABSTRACT (Continue on reverse if necessary and identify by block number) The representation of a nuclear space valued square integrable martingale by means of another nuclear space valued square integrable martingale is given in terms of stochastic integrals of operator valued processes. The construction of the stochastic integral goes through that of operator valued processes on Hilbert spaces. A new approach is given for the Hilbertian case, so that only the integration of Hilbert-Schmidt operator valued processes is needed to represent square integrable martingales.					
20. DISTRIBUTION/AVAILABILITY OF ABSTRACT UNCLASSIFIED/UNLIMITED <input checked="" type="checkbox"/> SAME AS RPT. <input type="checkbox"/> DTIC USERS <input type="checkbox"/>			21. ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED		
22a. NAME OF RESPONSIBLE INDIVIDUAL <i>My. Woodruff</i>			22b. TELEPHONE NUMBER (Include Area Code) <i>767-5027</i>		22c. OFFICE SYMBOL AFOSR/NM

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STOCHASTIC INTEGRATION FOR OPERATOR VALUED
PROCESSES ON HILBERT SPACES AND ON NUCLEAR SPACES

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Abstract : The representation of a nuclear space valued square integrable martingale by means of another nuclear space valued square integrable martingale is given in terms of stochastic integrals of operator valued processes. The construction of the stochastic integral goes through that of operator valued processes on Hilbert spaces. A new approach is given for the Hilbertian case, so that only the integration of Hilbert-Schmidt operator valued processes is needed to represent square integrable martingales.

Keywords : Stochastic integration, Hilbert space valued square integrable martingales, nuclear space valued square integrable martingales, representation of square integrable martingales.

* Research supported by AFOSR Contract No. F49620 82 C 0009.

Contract
F49620-85-C-
0144

INTRODUCTION

This paper deals with the integration of operator-valued stochastic processes with respect to Hilbert space and nuclear space valued square integrable martingales.

Generally speaking the stochastic integration for operator valued processes can be described as follows. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space with a filtration $(\mathcal{F}_t; t \in \mathbb{R}_+)$; (F, F') and (E, E') be two pairs of topological vector spaces in duality, and let M be an F' -valued square integrable martingale. We suppose, in this introduction, that the notion of F' (and E')-valued square integrable martingale and the space $L^2_E(\Omega, \mathcal{A}, \mathbb{P})$ of square integrable random variables with a locally convex topology are well defined. An elementary process with values in the space of continuous linear operators from F' into E' is defined as follows.

$$X = 1_{\{0\}} \otimes B_0 L_0 + \sum_{i=0}^n 1_{[t_i, t_{i+1})} \otimes B_i L_i$$

where $t_0 = 0 < t_1 < t_2 < \dots < t_{n+1}$, $B_i \in \mathcal{F}_{t_i}$ and L_i is a continuous linear operator from F' into E' . Then the integral of X with respect to M can be defined as the usual Stieltjes integral

$$\int_{\mathbb{R}_+} X_t dM_t = 1_{B_0} L_0 M_0 + \sum_{i=0}^n 1_{B_i} L_i (M_{t_{i+1}} - M_{t_i})$$

This quantity defines an E' -valued random variable. The main problem of the stochastic integration with respect to the given F' -valued square integrable martingale M is the definition of a topological vector space, say S^2 , of random functions with values in the space of linear operators from a certain space into E' , generated by the vector space V of elementary processes as given above so that the mapping $X \mapsto \int_{\mathbb{R}_+} X_t dM_t$ defines an algebraic and topological isomorphism

of V into $L^2_E(\Omega, \mathcal{A}, \mathbb{P})$. The stochastic integral of an arbitrary element of S^2 would then be defined by the extension of this isomorphism to S^2 . Partial stochastic integrals would also give F' -valued square integrable martingales.

The stochastic integral for operator valued processes on Hilbert spaces with respect to a Brownian motion was considered by Curtain and Falb [3], Daletskii [4], Gaveau [8], Kuo [15], Lepingle and Ouvrard [16], Yor [31], etc ; (cf also [5], [6] and [19] for definitions, applications and references) ; and the stochastic integral with respect to an arbitrary Hilbert space valued square integrable martingale by Kunita [14]. But the first extensive definition of the stochastic integral in the last case is due to Métivier and Pistone [20] who showed that processes in S^2 may take values in a space of noncontinuous operators. Important applications of Métivier and Pistone's approach were made by Pardoux for the study of stochastic partial differential equations [24], by Ouvrard for the representation of martingales [22] and for the linear filtering of Hilbert space valued systems [23], and by Martias for the derivation of the nonlinear filtering equation for Hilbert space valued semimartingales [17], [18].

In a work [12] dealing with the derivation of the nonlinear filtering equation related to a Hilbert space valued nonlinear system driven by Brownian motions, the first author showed the equivalence between stochastic integrals with respect to a Hilbert space valued Brownian motion in the sense of Métivier and Pistone and stochastic integrals with respect to the corresponding cylindrical Brownian motion. He also obtained the nonlinear filtering equation in terms of the corresponding cylindrical innovation process where the covariance operator and its pseudo-inverse do not appear explicitly as in [17], [18] and [23]. In [13] we extended the same method to the case of distribution valued nonlinear

systems driven by distribution valued Brownian motions. For this case, a construction of the stochastic integral can also be found in Itô's work [9]. In [12] the passage from a Brownian motion M to a cylindrical Brownian motion reposes on the fact that, in this case, Q is constant and $d\langle M \rangle = \alpha dt$ with a constant α . The introduction of adequate predictable fields of Hilbert spaces enables the extension of the same method to the case of arbitrary square integrable martingales.

In Section 1 we consider the stochastic integration with respect to a Hilbert space valued square integrable martingale and propose a method of constructing stochastic integrals and show that the set of martingales obtained by this method is the same as that obtained by Métivier and Pistone's method. Our method leads to some interesting algebraic and topological isomorphisms useful for the construction of stochastic integrals and representations of square integrable martingales.

In Section 2 we deal with the extension of the method to nuclear space valued square integrable martingales as defined by Ustunel in [29]. The extension is based on the fact that in the setting of [29], every nuclear space valued square integrable martingale M has almost all of its trajectories in a Hilbert space H . But in general H is not unique. Although the increasing process $\langle M \rangle$ corresponding to a martingale M can be defined independently of H , the representation $d\langle M \rangle = Q d\langle M \rangle$ depends on H . The entire approach uses a particular representation, but the set of all square integrable martingales obtained by the method developed here does not depend on the chosen Hilbert space H .

We end the paper with a short section of examples and applications where we consider the white noise process as a distribution valued one corresponding to the classical definition of the white noise as the derivative of the Brownian

motion in the sense of distributions. We show that the stochastic integral with respect to the white noise process is equivalent to the classical Ito integral.

1. STOCHASTIC INTEGRATION WITH RESPECT TO HILBERT SPACE VALUED MARTINGALES.

1.1 PRELIMINARIES

All the Hilbert spaces considered here are real. H and K represent separable Hilbert spaces. Scalar products and norms are denoted by (\cdot, \cdot) and $\|\cdot\|$, respectively. In order to point out, if necessary, the space on which they are defined, they will be indexed by the symbol representing the space.

$L(H, K)$ is the space of bounded linear operators from H into K with uniform norm $\|\cdot\|$, also denoted by $\|\cdot\|_{HK}$ for more precision. $L^1(H, K)$ is the space of nuclear operators with the trace norm $\|\cdot\|_1$ or $\text{tr}(\cdot)$, also denoted $\|\cdot\|_{H1K}$ if necessary and $L^2(H, K)$ is the space of Hilbert-Schmidt operators with the Hilbert-Schmidt norm $\|\cdot\|_2$, written also as $\|\cdot\|_{H2K}$. $H \hat{\otimes}_1 K$ (resp. $H \hat{\otimes}_2 K$) is the projective (resp. Hilbertian) tensor product of H with K . For notational conveniences, $H \hat{\otimes}_1 K$ (resp. $H \hat{\otimes}_2 K$) are identified with $L^1(H, K)$ (resp. $L^2(H, K)$) under the isometry which puts $h \otimes k$ into one-to-one correspondence with $(\cdot, h)_{HK}$ for $h \in H$, $k \in K$.

Unless the contrary is mentioned, we identify Hilbert spaces with their topological duals. We shall construct many pre-Hilbert spaces becoming Hilbert spaces after being divided by an equivalence relation. In order to simplify the presentation and the notations we shall identify this kind of pre-Hilbert spaces with the corresponding quotient Hilbert spaces.

The transpose of a linear operator A is denoted by A^* , its domain by $\text{Dom } A$ and range by $\text{Rg } A$. The closure of a set S is denoted by \overline{S} .

The algebraic and topological isomorphism between two topological vector spaces E and F is indicated as $E \cong F$.

For the terminology of stochastic analysis used here we refer to [7] and [19].

All random variables and processes are supposed to be defined on a complete probability space (Ω, \mathcal{A}, P) with a filtration $F = (F_t; t \in \mathbb{R}_+)$ satisfying the usual completeness and right continuity conditions. We take $\mathcal{A} = F_\infty$. Adapted processes, predictable processes, martingales, etc. are with respect to F . \mathcal{P} will represent the σ -algebra of predictable sets of $\mathcal{B}_+ \otimes F_\infty$ where \mathcal{B}_+ is the Borel σ -algebra of \mathbb{R}_+ . Ω' stands for $\mathbb{R}_+ \times \Omega$.

The equality between separable Hilbert space valued martingales is an equality up to an evanescent set in Ω' . Such martingales are defined as those having cadlag (right continuous and left limited) trajectories, (except on an evanescent set). For two square-integrable martingales M and N , with values in H and K , respectively, there is a unique $H \hat{\otimes}_1 K$ -valued cadlag process with integrable variation, denoted by $\langle M, N \rangle$ and called the oblique bracket of (M, N) , such that $M \otimes N - \langle M, N \rangle$ is a $H \hat{\otimes}_1 K$ -valued martingale, vanishing at $t=0$. The bracket $\langle M, M \rangle$ that we denote by $\langle M \rangle$ is called the increasing process of M . We put $\|M\|_t := \|\langle M \rangle_t\|_1$. This process is the unique real predictable increasing process with integrable variation for which $\|M\|^2 - \langle M \rangle$ is a martingale vanishing at $t=0$. We shall denote by $M^2(H)$ (resp. $M^2_c(H)$) the space of all H -valued square integrable (resp. continuous) martingales.

From now on M will represent a given martingale in $M^2(H)$ and λ will denote the measure defined on \mathcal{P} by $dP d\langle M \rangle$. There exists a predictable process Q with values in the cone of symmetric and nonnegative elements of $L^1(H, H)$, unique up

to a λ -equivalence, such that $||Q||_1 = 1$ λ -a.e. and $\langle M \rangle_t^* = \int_{0-}^t Q_s d\langle M \rangle_s$.

In what follows in this first paragraph, D represents a $L^2(H,H)$ -valued predictable process such that $DD^* = Q$ λ -a.e. An example to such a process is the positive square root of Q . Let $L^{*2}(D,H,K)$ be the space of processes X defined as follows : for λ -almost all $(t,\omega) \in \Omega'$, $X_t(\omega)$ is a (not necessarily continuous) linear operator from H into K such that $Rg D_t(\omega) \subset \text{Dom } X_t(\omega)$, $(X_t \circ D_t ; t \in \mathbb{R}_+)$ is a predictable process with values in $L^2(H,K)$ and $\int_{\Omega} ||X_t(\omega) \circ D_t(\omega)||_2^2 \lambda(dt,d\omega) < \infty$. The space $L^{*2}(D,H,K)$ is complete under the Hilbertian seminorm $X \mapsto [\int_{\Omega} ||X_t(\omega) \circ D_t(\omega)||_2^2 \lambda(dt,d\omega)]^{1/2}$, (cf. [19]). $\Lambda^2(D,H,K)$ will denote the Hilbert subspace of L^{*2} generated by all $L(H,K)$ -valued processes X in L^{*2} such that $(X_t h, k)_K$ is a real predictable process for all h in H and k in K . The set of all $L(H,K)$ -valued elementary processes of type :

$$(1.1.1) \quad X = 1_{(0)} \otimes B_0 A_0 + \sum_{i=0}^n 1_{[t_i, t_{i+1})} \otimes B_i A_i$$

where $0 = t_0 < t_1 < t_2 \dots$, $A_i \in L(H,K)$ and $B_i \in F_{t_i}$, is dense in $\Lambda^2(D,H,K)$.

The space $\Lambda^2(D,H,K)$ was introduced, with $D = Q^{1/2}$, by Métivier and Pistone in [20] for the definition of the stochastic integral with respect to M . We shall denote $\Lambda^2(D,H,\mathbb{R})$ by $\Lambda^2(D,H)$.

The stochastic integral of an elementary process of type (1.1.1) is the ordinary Stieltjes integral :

$$(1.1.2) \quad \int_{\mathbb{R}_+} X_t dM_t = 1_{B_0} A_0(M_0) + \sum_{i=0}^n 1_{B_i} A_i(M_{t_{i+1}} - M_{t_i}).$$

As $||X||_{\Lambda^2}^2 = E ||\int_{\mathbb{R}_+} X_t dM_t||_K^2$, the integral extends to $\Lambda^2(D,H,K)$ by isometry.

For each $X \in \Lambda^2(D,H,K)$, $\int_{0-}^t X dM$ defines an element of $M^2(K)$, denoted by $X.M$.

Before ending this introductory paragraph we just add here the fact that if $X \in \Lambda^2(D, H, K)$ and $Y \in \Lambda^2(D, H, G)$, where G is a separable Hilbert space, then

$$(1.1.3) \quad \langle X.M, Y.M \rangle_t = \int_{0-}^t (Y_s \circ D_s) \circ (X_s \circ D_s)^* d\langle M \rangle_s.$$

1.2 PREDICTABLE FIELDS OF HILBERT SPACES

We deal here again with the H -valued square integrable martingale M , the operator Q and the measure λ corresponding to it. All the operations carried on Q are valid λ -a.e., and in order to shorten the expressions we shall often omit the mention λ -a.e.. Let again D be a $L^2(H, H)$ -valued process such that $Q = D \circ D^*$. Define $L^{*2}(D_t(\omega), H)$ as the Hilbert space of linear operators f from H into H such that $Rg D_t(\omega) \subset \text{Dom } f$ and that $f \circ D_t(\omega)$ is continuous, with the scalar product $(f, g) = ((f \circ D_t(\omega))^*, (g \circ D_t(\omega))^*)_H$. Let us denote by $\tilde{H}_t(\omega)$ the Hilbert subspace of $L^{*2}(D_t(\omega), H)$ generated by H . The restriction to H of the scalar product on $\tilde{H}_t(\omega)$ is given by

$$(1.2.1) \quad (f, g)_{\tilde{H}_t(\omega)} = (D_t^*(\omega)f, D_t^*(\omega)g)_H = (Q_t(\omega)f, g)_H$$

Therefore, $D_t^*(\omega)$ defines an isometry from H into H which extends to an isometry from $\tilde{H}_t(\omega)$ into H . We denote this isometry by $I_t(\omega)$. We remark that if a sequence $(h_{n,t}(\omega); n \in \mathbb{N})$ is a CONS in $\tilde{H}_t(\omega)$, then its isometric image $(I_t(\omega) h_{n,t}(\omega); n \in \mathbb{N})$ is a CONS in $I_t(\omega)\tilde{H}_t(\omega) = \overline{D_t^*(\omega)H}$.

Our aim is to identify the processes in $\Lambda^2(D, H)$ with processes X such that for λ -a.e. $(t, \omega) \in \Omega^-$, $X_t(\omega) \in \tilde{H}_t(\omega)$. This brings us to look for a suitable field of Hilbert spaces in $\prod_{\Omega} \tilde{H}_t(\omega)$. We refer to [21] for details on Hilbertian fields. We briefly recall the definition.

DEFINITION 1.2.1. A predictable field of Hilbert spaces in $\Pi_{\Omega} \tilde{H}_t(\omega)$ is a vector subspace E having the following properties.

- (i) For all f and g in E , the mapping $(t, \omega) \rightarrow (f_t(\omega), g_t(\omega))_{\tilde{H}_t(\omega)}$ defines a predictable real process.
- (ii) Any $h \in \Pi_{\Omega} \tilde{H}_t(\omega)$ such that $(h_t(\omega), f_t(\omega))_{\tilde{H}_t(\omega)}$ is predictable for all $f \in E$ belongs to E .
- (iii) There exists a sequence $f = (f_n; n \in \mathbb{N})$ in E such that the sequence $f_t(\omega) = (f_{n,t}(\omega); n \in \mathbb{N})$ generates $\tilde{H}_t(\omega)$.

We choose an arbitrary linearly independent sequence $e = (e_n; n \in \mathbb{N})$ generating H and consider the sequence $\tilde{e}_t(\omega) = (\tilde{e}_{n,t}(\omega); n \in \mathbb{N})$ defined λ -a.e. recursively as follows.

$$(1.2.2) \left\{ \begin{array}{l} e_{0,t}(\omega) := e_0 \text{ and } e_{n,t}(\omega) := e_n - \sum_{k=0}^{n-1} (e_n, \tilde{e}_{k,t}(\omega))_{\tilde{H}_t(\omega)} \tilde{e}_{k,t}(\omega) \text{ for } n \geq 1, \\ \text{with} \\ \tilde{e}_{n,t}(\omega) := e_{n,t}(\omega) / \|e_{n,t}(\omega)\|_{\tilde{H}_t(\omega)} \text{ if } \|e_{n,t}(\omega)\|_{\tilde{H}_t(\omega)} > 0 \\ \quad = 0 \text{ if } \|e_{n,t}(\omega)\|_{\tilde{H}_t(\omega)} = 0. \end{array} \right.$$

The nonvanishing terms of $\tilde{e}_t(\omega)$ form a CONS in $\tilde{H}_t(\omega)$.

The sequence \tilde{e} will play the same role as f in the property (iii) of the above definition. Before giving the characterization of the predictable field we shall be using here, we consider some predictable processes playing an important role in what follows.

Q is a predictable process, i.e. for all $f, g \in H$, $(Q_t(\omega)f, g)_H = (f, g)_{\tilde{H}_t(\omega)}$ is predictable. Therefore, by construction, all the \tilde{e}_n 's are predictable. Moreover Q has the following representation (as a process with values in $L^2(H, H) \supset L^1(H, H)$).

$$Q_t(\omega) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \mu_{m,n,t}(\omega) h_m \otimes h_n$$

where $(h_n; n \in \mathbb{N})$ is a CONS in H , the coefficients $\mu_{m,n}$ are predictable,

$\sum_m \sum_n \mu_{m,n,t}^2(\omega) < \infty$, $\mu_{m,m} \geq 0$ and $\mu_{m,n} = \mu_{n,m}$. If X is an H -valued predict-

able process, then

$$Q_t(\omega)X_t(\omega) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \mu_{m,n,t}(\omega)(X_t(\omega), h_m)_H h_n.$$

Since the processes $(X_t(\omega), h_m)_H$ are predictable, QX is a predictable process.

Therefore, for all $h \in H$, $(X_t(\omega), h)_{\tilde{H}_t(\omega)} = (Q_t(\omega)X_t(\omega), h)_H$ defines a predictable

process. We have in $\tilde{H}_t(\omega)$

$$(1.2.3) \quad \tilde{e}_{n,t}(\omega) = \sum_{j=0}^n u_{n,j,t}(\omega) e_j$$

with predictable coefficients. We deduce from this that for any H -valued

predictable process X , $(X_t(\omega), \tilde{e}_{m,t}(\omega))_{\tilde{H}_t(\omega)}$ is a predictable process.

If $X \in \Pi_{\Omega'} \tilde{H}_t(\omega)$, then it has the following representation.

$$(1.2.4) \quad X_t(\omega) = \sum_{n=0}^{\infty} a_{n,t}(\omega) \tilde{e}_{n,t}(\omega), \quad \lambda\text{-a.e.}$$

where the series converges in $\tilde{H}_t(\omega)$ and $\sum_{n=0}^{\infty} a_{n,t}^2(\omega) < \infty$. It is obvious that an

H -valued predictable process X belongs to $\Pi_{\Omega'} \tilde{H}_t(\omega)$ and if it is represented

by the series (1.2.4) then the coefficients are predictable.

We denote by S the set of all H -valued predictable processes. By using the representation (1.2.4) of elements of $\Pi_{\Omega'} \tilde{H}_t(\omega)$ we immediately see that, for an element $X \in \Pi_{\Omega'} \tilde{H}_t(\omega)$, $(X_t(\omega), Y_t(\omega))_{\tilde{H}_t(\omega)}$ is predictable for all $Y \in S$ iff

$(X_t(\omega), \tilde{e}_{n,t}(\omega))_{H_t(\omega)}$ is predictable for all $n \in \mathbb{N}$. This shows that the measurability of $(X_t(\omega), \tilde{e}_{n,t}(\omega))_{H_t(\omega)}$ for all $Y \in S$ does not depend on the chosen particular sequence \tilde{e} .

Moreover, if $(X_t(\omega), h)_{H_t(\omega)}$ is predictable for all $h \in H$, then (1.2.3) implies that $(X_t(\omega), \tilde{e}_{n,t}(\omega))_{H_t(\omega)}$ is predictable for all n . Conversely, if this is the case, $(X_t(\omega), h)_{H_t(\omega)}$ is predictable for all $h \in H$, because any constant process with values in H is a particular H -valued predictable process.

We denote by $E(D, H)$ the vector subspace of $\prod_{\Omega^-} \tilde{H}_t(\omega)$ consisting of all elements X such that $(X_t(\omega), \tilde{e}_{n,t}(\omega))_{H_t(\omega)}$ is predictable for all $n \in \mathbb{N}$. This is the set of all elements X such that $(X_t(\omega), Y_t(\omega))_{H_t(\omega)}$ is predictable for all $Y \in S$. Therefore, $E(D, H)$ does not depend on \tilde{e} . $E(D, H)$ is a predictable field of Hilbert spaces and said to be generated by S . We remark that $E(D, H)$ is the set of all elements X such that $(X_t(\omega), h)_{H_t(\omega)}$ is predictable for all $h \in H$. The elements of $E(D, H)$ are called predictable fields of vectors.

$\tilde{\Lambda}^2(D, H)$ will be the Hilbert space of all elements X of $E(D, H)$ such that

$$(1.2.5) \quad ||X||_{\tilde{\Lambda}^2}^2 = \int_{\Omega^-} ||X_t(\omega)||_{H_t(\omega)}^2 \lambda(dt, d\omega) < \infty.$$

$\tilde{\Lambda}^2(D, H)$ is the space of all elements X of $E(D, H)$ whose representation in terms of \tilde{e} as in (1.2.4) is such that

$$(1.2.6) \quad ||X||_{\tilde{\Lambda}^2}^2 = \sum_{n=0}^{\infty} \int_{\Omega^-} a_{n,t}^2(\omega) \lambda(dt, d\omega) < \infty.$$

We shall show that $\tilde{\Lambda}^2(D, H)$ is isometric to $\Lambda^2(D, H)$ and obtain an equivalent definition of stochastic integrals with respect to M . But we need to show this equivalence especially for operator-valued processes. For this purpose we first define, in the same way as $E(D, H)$, a predictable field of Hilbert spaces consisting of Hilbert-Schmidt operators.

K represents here another separable Hilbert space and $(k_n; n \in \mathbb{N})$ a CONS in K . We consider the vector subspace $E(D, H, K)$ of $\Pi_{\Omega}(\tilde{H}_t(\omega) \hat{\otimes}_2 K)$ consisting of all elements X for which $(X_t(\omega), \tilde{e}_{m,t}(\omega) \hat{\otimes}_2 k_n)_{\tilde{H}_t(\omega) \hat{\otimes}_2 K}$ is predictable for all $(m, n) \in \mathbb{N}^2$. If $X \in \Pi_{\Omega}(\tilde{H}_t(\omega) \hat{\otimes}_2 K)$, then it has the representation

$$(1.2.7) \quad X_t(\omega) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{m,n,t}(\omega) \tilde{e}_{m,t}(\omega) \hat{\otimes}_2 k_n$$

where the series converges in $\tilde{H}_t(\omega) \hat{\otimes}_2 K$, i.e. $\sum_m \sum_n b_{m,n,t}^2(\omega) < \infty$. Therefore,

$X \in E(D, H, K)$ iff the coefficients $b_{m,n}$ are predictable processes. We then see that $E(D, H, K)$ is a predictable field of Hilbert spaces in $\Pi_{\Omega}(\tilde{H}_t(\omega) \hat{\otimes}_2 K)$.

By representing $X_t(\omega)$ as a Hilbert-Schmidt operator from $\tilde{H}_t(\omega)$ into K , we can write $(X_t(\omega), \tilde{e}_{m,t}(\omega) \hat{\otimes}_2 k_n)_2 = (X_t(\omega) \tilde{e}_{m,t}(\omega), k_n)_K$. We then see that this defines a predictable real process, for fixed m and for all $n \in \mathbb{N}$, iff $(X_t(\omega) \tilde{e}_{m,t}(\omega), k)_K$ is predictable for all $k \in K$ or equivalently iff $X_t(\omega) \tilde{e}_{m,t}(\omega)$ is a K -valued predictable process. Therefore $X \in E(D, H, K)$ iff $(X_t(\omega), \tilde{e}_{m,t}(\omega) \hat{\otimes}_2 k)_2 = (X_t(\omega) \tilde{e}_{m,t}(\omega), k)_K = (\tilde{e}_{m,t}(\omega), X_t^*(\omega)k)_{\tilde{H}_t(\omega)}$ is predictable for all $m \in \mathbb{N}$ and $k \in K$, or equivalently iff $X_t^*(\omega)k \in E(D, H)$ for all $k \in K$. We also have seen that $X_t^*(\omega)k \in E(D, H)$ iff $(Y_t(\omega), X_t^*(\omega)k)_{\tilde{H}_t(\omega)} = (X_t(\omega)Y_t(\omega), k)_K = (X_t(\omega), Y_t(\omega) \hat{\otimes}_2 k)_2$ is predictable for all $Y \in S$, or equivalently iff $(h, X_t^*(\omega)k)_{\tilde{H}_t(\omega)} = (X_t(\omega)h, k)_K = (X_t(\omega), h \hat{\otimes}_2 k)_2$ is predictable for all $h \in H$.

Therefore, we see that the definition of $E(D, H, K)$ does not depend on the chosen sequences $(e_n; n \in \mathbb{N}) \subset H$ and $(k_n; n \in \mathbb{N}) \subset K$ and that $E(D, H, K)$ is generated by $(Y_t(\omega) \otimes k; Y \in S, k \in K)$ as well as by constant processes with values in $H \otimes K$, (the tensor product is taken on $\tilde{H}_t(\omega) \hat{\otimes}_2 K$).

Now we define $\tilde{\Lambda}^2(D, H, K)$ as the Hilbert space of all elements X in $E(D, H, K)$ such that

$$(1.2.8) \quad ||X||_{\tilde{\Lambda}^2}^2 = \int_{\Omega} ||X_t(\omega)||_{\tilde{H}_t(\omega)}^2 dK^\lambda(dt, d\omega) < \infty$$

we see that $\tilde{\Lambda}^2(D, H, K)$ consists of those elements X having the representation (1.2.7) with predictable coefficients such that

$$(1.2.9) \quad ||X||_{\tilde{\Lambda}^2}^2 = \sum_m \sum_n \int_{\Omega} b_{m,n,t}^2(\omega) d\lambda(dt, d\omega) < \infty$$

The construction of $\tilde{\Lambda}^2(D, H, K)$ suggests the following statement.

PROPOSITION 1.2.2. $\tilde{\Lambda}^2(D, H, K) \cong \tilde{\Lambda}^2(D, H) \hat{\otimes}_2 K$.

Proof : The set of elements of type

$$(1.2.10) \quad \tilde{X}_t(\omega) = \sum_{m=0}^M \sum_{n=0}^N b_{m,n,t}(\omega) \tilde{e}_{m,t}(\omega) \otimes k_n \in \tilde{H}_t(\omega) \hat{\otimes}_2 K$$

where $(k_n; n \in \mathbb{N})$ is a CONS in K and $\sum_{m=0}^M \sum_{n=0}^N \int_{\Omega} b_{m,n}^2 d\lambda < \infty$, is dense in $\tilde{\Lambda}^2(D, H, K)$.

Define $X' \in \tilde{\Lambda}^2(D, H) \hat{\otimes}_2 K$ by $\sum_{n=0}^N Y_n \otimes k_n$ where Y_n is the H -valued predictable process defined by $Y_{n,t}(\omega) = \sum_{m=0}^M b_{m,n,t}(\omega) \tilde{e}_{m,t}(\omega)$. We see that $||X|| = ||X'||$.

The mapping $\tilde{X} \rightarrow X'$ extends to an isometry I_1 of $\tilde{\Lambda}^2(D, H, K)$ into $\tilde{\Lambda}^2(D, H) \hat{\otimes}_2 K$.

Conversely, if $Y \in \tilde{\Lambda}^2(D, H)$ is represented by $Y_t(\omega) = \sum_{m=0}^{\infty} a_{m,t}(\omega) \tilde{e}_{m,t}(\omega)$ and if

$X' = Y \otimes k \in \tilde{\Lambda}^2(D, H) \hat{\otimes}_2 K$, we define $\tilde{X} \in \tilde{\Lambda}^2(D, H, K)$ by $\tilde{X}_t(\omega) = \sum_{m=0}^{\infty} a_{m,t}(\omega) (\tilde{e}_{m,t}(\omega) \otimes k)$.

We see that $||X'|| = ||\tilde{X}||$. The mapping $X' \rightarrow \tilde{X}$ then extends to an isometry I_2 from $\tilde{\Lambda}^2(D, H) \hat{\otimes}_2 K$ into $\tilde{\Lambda}^2(D, H, K)$. Since $I_1 I_2 Y \hat{\otimes} k = Y \hat{\otimes} k \in \tilde{\Lambda}^2(D, H) \hat{\otimes}_2 K$, I_1 is the inverse of I_2 and the two isometries are onto. \square

According to the above proof if $X' \in \tilde{\Lambda}^2(D, H) \hat{\otimes}_2 K$ is represented by

$$X' = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{m,n} X_m \hat{\otimes} k_n$$

where the coefficients are real and $\sum_m \sum_n c_{m,n}^2 < \infty$, $(X_n; n \in \mathbb{N}) \subset \tilde{\Lambda}^2(D, H)$ and $(k_n; n \in \mathbb{N}) \subset K$ are orthonormal sequences, then the isometric image \tilde{X} of X' in $\tilde{\Lambda}^2(D, H, K)$ can be represented by

$$X_t(\omega) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{m,n} X_{m,t}(\omega) \hat{\otimes} k_n \quad \lambda\text{-a.e.}$$

where the tensor products are taken in $\tilde{H}_t(\omega) \hat{\otimes}_2 K$ and the series converges in $\tilde{\Lambda}^2(D, H, K)$.

The following theorem will allow a definition of stochastic integrals, equivalent to that of Métivier and Pistone in [20].

THEOREM 1.2.3. $\tilde{\Lambda}^2(D, H, K) \cong \Lambda^2(D, H, K)$.

Proof : $(k_n; n \in \mathbb{N})$ is again a CONS in K . Let $\tilde{X} \in \tilde{\Lambda}^2(D, H, K)$ be given by (1.2.10) and let us define X by

$$(1.2.11) \quad X_t(\omega) = \sum_{m=0}^M \sum_{n=0}^N b_{m,n,t}(\omega) \tilde{e}_{m,t}(\omega) \hat{\otimes} k_n \in H \hat{\otimes}_2 K$$

where the tensor product is taken in $H \hat{\otimes}_2 K$. X is predictable and

$$X_t(\omega) \circ D_t(\omega) = \sum_{m=0}^M \sum_{n=0}^N b_{m,n,t}(\omega) (D_t^*(\omega) \tilde{e}_{m,t}(\omega)) \otimes k_n$$

Since $(D_t^*(\omega) \tilde{e}_{m,t}(\omega); m \in \mathbb{N})$ is a CONS in $I_t(\omega) \tilde{H}_t(\omega)$, this gives us the representation of the Hilbert-Schmidt operator $X_t(\omega) \circ D_t(\omega)$ in terms of an orthonormal sequence in $H \otimes_2 K$. We then have

$$\begin{aligned} \|X\|_{\Lambda^2}^2 &= \int_{\Omega} \|D_t^*(\omega) \circ X_t^*(\omega)\|_{K \otimes H}^2 \lambda(dt, d\omega) \\ &= \sum_{m=0}^M \sum_{n=0}^N \int_{\Omega} b_{m,n,t}^2(\omega) \lambda(dt, d\omega) = \|\tilde{X}\|_{\Lambda^2}^2. \end{aligned}$$

Therefore, the mapping $\tilde{X} \rightarrow X$ extends to an isometry I_1 of $\tilde{\Lambda}^2(D, H, K)$ into $\Lambda^2(D, H, K)$.

Conversely, let $X \in \Lambda^2(D, H, K)$ be $L(H, K)$ -valued. Then λ -a.e. $X_t(\omega) \circ D_t(\omega)$ is a Hilbert-Schmidt operator from H into K that we can represent by the following series.

$$\begin{aligned} (1.2.12) \quad X_t(\omega) \circ D_t(\omega) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} d_{m,n,t}(\omega) (D_t^*(\omega) \tilde{e}_{m,t}(\omega)) \otimes k_n \\ &\quad + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{m,n,t}(\omega) h_m \otimes k_n \end{aligned}$$

where $(h_m; m \in \mathbb{N})$ is a CONS in the orthogonal complement of $I_t(\omega) \tilde{H}_t(\omega)$ in H and of course the coefficients are predictable. But we have $c_{m,n,t}(\omega) = 0$ for all m, n ; because, for all $h \in H$, $(h_m, D_t^*(\omega) h)_H = 0$, i.e. $D_t(\omega) h_m = 0$, and hence $c_{m,n,t}(\omega) = (X_t(\omega) \circ D_t(\omega) h_m, k_n)_K = 0$. Therefore, we have $\|X_t(\omega) \circ D_t(\omega)\|_{H \otimes K}^2 =$

$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} d_{m,n,t}^2(\omega)$. Now, let us define \tilde{X} by

$$(1.2.13) \quad \tilde{X}_t(\omega) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} d_{m,n,t}(\omega) \tilde{e}_{m,t}(\omega) \otimes k_n$$

as an element of $\tilde{H}_t(\omega) \hat{\otimes}_2 K$. We see that $\|X\|_{\Lambda^2} = \|\tilde{X}\|_{\tilde{\Lambda}^2}$. Therefore the mapping $X \rightarrow \tilde{X}$ defines an isometry I_2 from $\Lambda^2(D, H, K)$ into $\tilde{\Lambda}^2(D, H, K)$.

Now we start from the element $\tilde{X} \in \tilde{\Lambda}^2(D, H, K)$ defined by (1.2.10), obtain $I_1 \tilde{X} = X$ given by (1.2.11) and get from this

$$\begin{aligned} X_t(\omega) \circ D_t(\omega) &= (I_1 \tilde{X})_t(\omega) \circ D_t(\omega) \\ &= \sum_{m=0}^M \sum_{n=0}^N b_{m,n,t}(\omega) (D_t^*(\omega) \tilde{e}_{m,t}(\omega)) \otimes k_n \end{aligned}$$

where the tensor product is taken in $H \hat{\otimes}_2 K$. We see that we have $I_2 \circ I_1 \tilde{X} = \tilde{X}$. Therefore $I_2 \circ I_1$ is the identity on $\tilde{\Lambda}^2(D, H, K)$. This proves that the two isometries are onto and $I_1 = I_2^{-1}$. \square

We denote by J the isometry of $\tilde{\Lambda}^2(D, H, K)$ onto $\Lambda^2(D, H, K)$. As a result of the above proof we also get the following proposition.

PROPOSITION 1.2.4. The set of all processes of the form (1.2.11) where the tensor product is taken on $H \hat{\otimes}_2 K$ is dense in $\Lambda^2(D, H, K)$.

Theorem 1.2.3 and Proposition 1.2.2 also give the following result :

PROPOSITION 1.2.5. $\Lambda^2(D, H, K) \cong \Lambda^2(D, H) \hat{\otimes}_2 K$.

REMARK 1.2.6. As a particular case, Theorem 1.2.3 says that $\tilde{\Lambda}^2(D, H)$ and $\Lambda^2(D, H)$ are isometric. If $\tilde{X} \in \tilde{\Lambda}^2(D, H)$ is represented by

$$(1.2.14) \quad \tilde{X}_t(\omega) = \sum_{n=0}^N a_{n,t}(\omega) \tilde{e}_{n,t}(\omega)$$

then, as we have seen in the proof of Theorem 1.2.3, we have

$$J\tilde{X} = X = \tilde{X} \in \Lambda^2(D, H).$$

Therefore, J is an isometric injection on the set of all processes of type (1.2.14). We have even more. If X is an H -valued predictable process such that $\int_{\Omega} ||D_t^*(\omega)X_t(\omega)||_H^2 \lambda(dt, d\omega) < \infty$ then it belongs to both $\tilde{\Lambda}^2(D, H)$ and $\Lambda^2(D, H)$ and has the same norm in both of them. The set of all processes of this type generates the two spaces. We see that $\tilde{\Lambda}^2(D, H)$ and $\Lambda^2(D, H)$ only differ on the completion of their vector subspace of all H -valued predictable processes.

1.3. STOCHASTIC INTEGRALS

We continue to use the notations of Paragraph 1.2. In Paragraph 1.1 we have seen the definition of the stochastic integral for a process in $\Lambda^2(D, H)$. We adopt the same definition for all H -valued processes in $\Lambda^2(D, H)$. We have seen that they also are in $\tilde{\Lambda}^2(D, H)$. If X is such a process, then $E(\int_{\mathbb{R}_+} X_t dM_t)^2 = ||X||_{\Lambda^2}^2 = ||X||_{\tilde{\Lambda}^2}^2$. The stochastic integral of \tilde{e}_n is then well defined and we put

$$(1.3.1) \quad M_{n,t} := \int_{0-}^t \tilde{e}_{n,s} dM_s = (\tilde{e}_n \cdot M)_t$$

We remark that

$$(1.3.2) \quad \begin{aligned} \langle M_m, M_n \rangle_t &= \int_{0-}^t (\tilde{e}_{m,s}(\omega), \tilde{e}_{n,s}(\omega))_{\tilde{H}_s(\omega)} d\langle M \rangle_s(\omega). \\ &= \langle M \rangle_t \quad \text{if } n = m, \quad 0 \quad \text{if } n \neq m. \end{aligned}$$

Hence $(M_n; n \in \mathbb{N})$ is a sequence of strongly orthogonal martingales.

Since the set of processes of type (1.2.14) are dense in $\tilde{\Lambda}^2(D, H)$ and $\Lambda^2(D, H)$ the stochastic integral of an arbitrary element $\tilde{X} \in \tilde{\Lambda}^2(D, H)$ is defined as the limit in $L^2(\Omega, \mathcal{A}, \mathbb{P})$ of the integrals of an approximating sequence of type (1.2.14).

Consequently, if $\tilde{X} \in \tilde{\Lambda}^2(D, H)$ is represented by (1.2.4) we define

$$(1.3.3) \quad \int_{\mathbb{R}_+} \tilde{X}_t dM_t = \sum_{n=0}^{\infty} \int_{\mathbb{R}_+} a_{n,t} dM_{n,t}$$

with the series converging in $L^2(\Omega, \mathcal{A}, \mathbb{P})$. Obviously, the random variable defined by (1.3.3) coincides with the stochastic integral of $X = J\tilde{X}$ in the sense of Métivier and Pistone.

For an element $\tilde{X} \in \tilde{\Lambda}^2(D, H, K)$ having the representation (1.2.7) with the norm given by (1.2.9) we define the stochastic integral by

$$(1.3.4) \quad \int_{\mathbb{R}_+} \tilde{X}_t dM_t = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(\int_{\mathbb{R}_+} b_{m,n,t} dM_{m,t} \right) k_n$$

where the series converges in $L_K^2(\Omega, \mathcal{A}, \mathbb{P})$. We have

$$(1.3.5) \quad E \left\| \int_{\mathbb{R}_+} \tilde{X}_t dM_t \right\|_K^2 = \|\tilde{X}\|_{\tilde{\Lambda}}^2.$$

It is obvious that the value of the stochastic integral (1.3.4) does not depend on the particular representation of \tilde{X} .

The correspondence between this definition of stochastic integrals and that given by Métivier and Pistone is not straightforward as in the case of functional valued processes. The result is given in the following theorem.

THEOREM 1.3.1. Let $\tilde{X} \in \tilde{\Lambda}^2(D, H, K)$ and let $X = J\tilde{X} \in \Lambda^2(D, H, K)$. Then the stochastic integral of \tilde{X} in the sense of (1.3.4) and the stochastic integral of X in the sense of Métivier and Pistone coincide.

Proof : If \tilde{X} is given by (1.2.10), then $X = J\tilde{X}$ is given by (1.2.11). It is obvious that $\int_{\mathbb{R}_+} \tilde{X} dM$ in the sense of (1.3.4) and $\int_{\mathbb{R}_+} X dM$ in the sense of Métivier and Pistone coincide. Since the set of processes of type (1.2.10) and the set of

processes of type (1.2.11) are dense in $\tilde{\Lambda}^2(D, H, K)$ and $\Lambda^2(D, H, K)$, respectively, we obtain the statement of the theorem. \square

For a process $\tilde{X} \in \tilde{\Lambda}^2(D, H, K)$ we denote by $\tilde{X}.M$ the square-integrable martingale defined by $\int_{0-}^t \tilde{X}_s dM_s$. Theorem 1.3.1 says that $\tilde{X}.M = (J\tilde{X}).M$, the stochastic integrals of the right-hand side being defined in the sense of Métivier and Pistone. All along this paper the stochastic integrals are defined in the sense of (1.3.4). If ever a stochastic integral is defined in the sense of Métivier and Pistone we will always precise it.

PROPOSITION 1.3.2. Let $\tilde{X} \in \tilde{\Lambda}^2(D, H)$. Then

$$(1.3.6) \quad \langle \tilde{X}.M \rangle_t = \int_{0-}^t \|\tilde{X}_s(\omega)\|_{H_s(\omega)}^2 d\langle M \rangle_s(\omega) .$$

Proof : Let \tilde{X} (resp. \tilde{X}_N) be represented by the right hand side of (1.2.4) (resp. 1.2.14), then (\tilde{X}_N) converges to \tilde{X} in $\tilde{\Lambda}^2$ and $(X_N = J\tilde{X}_N = \tilde{X}_N; N \in \mathbb{N})$ converges to $X = J\tilde{X}$ in $\Lambda^2(D, H)$. Consequently, there is a subsequence $(\tilde{X}_{N_i}; i \in \mathbb{N})$ such that $\int_{0-}^t \|\tilde{X}_{N_i,s}(\omega)\|_{H_s(\omega)}^2 d\langle M \rangle_s(\omega)$ converges a.s. to $\int_{0-}^t \|\tilde{X}_s(\omega)\|_{H_s(\omega)}^2 d\langle M \rangle_s(\omega)$ and to $\langle X.M \rangle_t$. (In this last expression $X.M$ is computed by means of Métivier and Pistone's integrals). Since $\tilde{X}.M = X.M$ and hence $\langle \tilde{X}.M \rangle = \langle X.M \rangle$ with the integral $X.M$ computed in the sense of Métivier and Pistone, (1.3.6) holds. \square

Let G be another separable real Hilbert space. Then by approximating the elements of $\tilde{\Lambda}^2(D, H, K)$ and $\tilde{\Lambda}^2(D, H, G)$ by finite sums of type (1.2.10) we can similarly prove the following result.

PROPOSITION 1.3.3. Let $\tilde{X} \in \tilde{\Lambda}^2(D, H, K)$ and $\tilde{Y} \in \tilde{\Lambda}^2(D, H, G)$

then

$$(1.3.7) \quad \langle \tilde{X}.M, \tilde{Y}.M \rangle_t = \int_{0-}^t \tilde{Y}_s \circ \tilde{X}_s^* d\langle M \rangle_s .$$

We shall need in Section 2 the following result.

PROPOSITION 1.3.4. Let K and G be separable Hilbert spaces and A a continuous linear operator from K into G . Then for $X \in \tilde{\Lambda}^2(D, H, K)$ we have the following martingale equality

$$(1.3.8) \quad A \int_{0-}^t X_s \, dM_s = \int_{0-}^t A_0 X_s \, dM_s$$

Proof : It is easy to see that (1.3.8) holds for all processes X of type (1.2.10) in $\tilde{\Lambda}^2(D, H, K)$. Then the equality (1.3.8) is obtained for an arbitrary process X by the density of processes of type (1.2.10) in $\tilde{\Lambda}^2(D, H, K)$. \square

We reproduce here the following representation theorem proved in [22] with stochastic integrals defined in the sense of Métivier and Pistone. We have however seen in Theorem 1.3.1 that we obtain the same set of martingales with the integrals defined by (1.3.4).

THEOREM 1.3.5. Let M and N be square integrable martingales with values in H and K , respectively. Then there is a process $X \in \tilde{\Lambda}^2(D, H, K)$ such that

$$(1.3.9) \quad N = X.M + N^\perp$$

where N^\perp is an element of $M^2(K)$ orthogonal to M . This representation of N is unique in the sense that X is unique up to λ -equivalence and N^\perp is unique up to an evanescent set.

According to this Theorem M itself must have a representation by stochastic integrals. We express this representation in the following proposition.

PROPOSITION 1.3.6. We have

$$M_t = \int_{0-}^t (D_s \circ I_s) \, dM_s$$

where $I_t(\omega)$ is the isometry of $\tilde{H}_t(\omega)$ into H . If $(h_n; n \in \mathbb{N})$ is a CONS in H then we have $D_t(\omega) \circ I_t(\omega) = \sum_{n=0}^{\infty} h_n \otimes h_n \in \tilde{H}_t(\omega) \hat{\otimes}_2 H$.

Proof. Let $(h_n; n \in \mathbb{N})$ be a CONS in H . Then the identity operator on H can be formally represented by $i = \sum_{n=0}^{\infty} h_n \otimes h_n$. This means that each element $h \in H$ is represented by $h = \sum_n (h, h_n)_H h_n$. For $f, g \in H$ we have $(i \circ D_t(\omega)f, g)_H = (f, D_t^*(\omega) \circ i^*g)_H = \sum_n (h_n, g)_H (D_t^*(\omega)h_n, f)_H = ([\sum_n (D_t^*(\omega)h_n) \otimes h_n]f, g)_H$.

Therefore, $i \circ D_t(\omega)$ has the following representation

$i \circ D_t(\omega) = \sum_n (D_t^*(\omega)h_n) \otimes h_n \in H \hat{\otimes}_2 H$. According to the proof of Theorem 1.2.3. the isometric image of i is $X_t(\omega) = \sum_{n=0}^{\infty} h_n \otimes h_n \in \tilde{H}_t(\omega) \hat{\otimes}_2 H$.

For $f \in \tilde{H}_t(\omega)$ we can write

$$X_t(\omega)f = \sum_{n=0}^{\infty} (h_n, f)_{\tilde{H}_t(\omega)} h_n = \sum_{n=0}^{\infty} (I_t(\omega)h_n, I_t(\omega)f)_H h_n = \sum_{n=0}^{\infty} (h_n, D_t(\omega)I_t(\omega)f)_H h_n.$$

Therefore $X_t(\omega) = D_t(\omega) \circ I_t(\omega)$. According to Theorem 1.3.1 we have

$$M_t = \int_{0-}^t i \, dM_s = \int_{0-}^t (D_s \circ I_s) dM_s$$

where the first integral is taken in the sense of Métivier and Pistone and the second integral in the sense of (1.3.4). We also remark that the integral on the right hand side is equal to $\sum_n (M_t, h_n)_H h_n = M_t$. \square

REMARK 1.3.7. In our construction, the factorization $D_t(\omega) \circ D_t^*(\omega) = Q_t(\omega)$ played an important role in the definition of $\tilde{H}_t(\omega)$, because we wanted $\tilde{H}_t(\omega)$ to be identified with some space of linear functionals on H . For the definition of stochastic integrals this is not necessary. In fact, we can

define $\tilde{H}_t(\omega)$ as any (abstract) completion of H under the Hilbertian topology induced by $(f,g)_{\tilde{H}_t(\omega)} = (Q_t(\omega)f,g)_H$ without having to factorize $Q_t(\omega)$. The space $\tilde{\Lambda}^2$ obtained in this way is isometric with $\tilde{\Lambda}^2(D,H,K)$ constructed here and the definition of the stochastic integral that we could give would be the same. The role played by $D_t^*(\omega)$ would then be played by the isometry of the new $\tilde{H}_t(\omega)$ into H . But we still think that there is some advantage in visualizing $\tilde{H}_t(\omega)$ as a space of linear functionals on H .

1.4. AN EXAMPLE: BROWNIAN MOTION

The stochastic integrals in the sense of the preceding paragraph was defined in [12] for an H -valued Brownian motion. In this case the description of the space $\tilde{\Lambda}^2(D,H,K)$ is much simpler. In order to avoid localization problems we take $t \in [0,T]$, $T \in \mathbb{R}_+$.

An H -valued Brownian motion $W = (W_t; t \in [0,T])$ is a continuous square-integrable H -valued martingale such that $W_0 = 0$ and $\langle W \rangle_t = tQ$, where Q is a nonnegative symmetric element of $L^1(H,H)$, called the covariance operator of W . Hence $\langle W \rangle$ has the following representation: $\langle W \rangle_t = \int_0^t Q dt = \int_0^t \frac{Q}{\text{Tr} Q} (\text{Tr} Q) dt$, with $Q_t = Q/\text{Tr} Q$ and $\langle W \rangle_t = t\text{Tr} Q$. Instead of doing the factorization of Q_t we do the factorization of Q for the construction of \tilde{H} , so that the measure $\lambda(dt,d\omega)$ is replaced by $dt P(d\omega)$. We denote again by λ the corresponding measure. We consider then a factorization $Q = D^*$ of Q with $D \in L^2(H,H)$. Since D does not depend on (t,ω) , $\tilde{H}_t(\omega) = \tilde{H}$ for all $(t,\omega) \in [0,T] \times \Omega = \Omega'$. \tilde{H} is the completion of H under the scalar product $(f,g)_{\tilde{H}} = (D^* f, D^* g)_H$. We see that $\tilde{\Lambda}^2(D,H)$ coincides with $L^2_{\tilde{H}}(\Omega', P, \lambda)$ and $\tilde{\Lambda}^2(D,H,K)$ with $L^2_{\tilde{H}}(\Omega', P, \lambda)$.

Let us define the mapping $\tilde{W}: [0,T] \times \tilde{H} \rightarrow L^2(\Omega, \mathcal{A}, \mathbb{P})$ by $(t,h) \rightarrow \tilde{W}_t(h) = \int_0^t h dW$, this last integral being the stochastic integral of the constant

process in $L^2_H(\Omega, \mathcal{P}, \lambda)$ whose value is $h \in \tilde{H}$. $\tilde{W}_t(h) = (W_t, h)_H$ for $h \in H$. If $(h_n; n \in \mathbb{N}) \subset H$ is a sequence converging to h in \tilde{H} , then $\tilde{W}_t(h)$ is the L^2 -limit of the sequence $((h_n, W_t)_H; n \in \mathbb{N})$. \tilde{W} defines a standard cylindrical Brownian motion (cf. [19]), i.e. $(\tilde{W}_t(h) / \|h\|_{\tilde{H}}; t \in [0, T])$ is a real Brownian motion and for each t , $h \mapsto \tilde{W}_t(h)$ is a continuous linear mapping of \tilde{H} into $L^2(\Omega, \mathcal{A}, \mathbb{P})$ such that $\|\tilde{W}_t(h)\|^2 = t \|h\|_{\tilde{H}}^2$.

Since H is dense in \tilde{H} we can choose a CONS $(h_n; n \in \mathbb{N})$ in \tilde{H} contained in H . Then $W_{n,t} = \tilde{W}_t(h_n) = (W_t, h_n)_H$ defines a sequence of independent Brownian motions. If X is an element of $L^2_H(\Omega, \mathcal{P}, \lambda)$, it is represented by

$$(1.4.1) \quad X_t = \sum_{n=0}^{\infty} a_{n,t} h_n \quad \text{with} \quad \|X\|^2 = \sum_{n=0}^{\infty} \int_0^T E(a_{n,t}^2) dt$$

where $a_{n,t} = (X_t, h_n)_{\tilde{H}}$ is a predictable process. The stochastic integral of X with respect to W is then defined by

$$(1.4.2) \quad \int_0^T X_t dW_t = \sum_{n=0}^{\infty} \int_0^T a_{n,t} dW_{n,t},$$

where the series converges in $L^2(\Omega, \mathcal{A}, \mathbb{P})$.

Similarly, if $(k_n; n \in \mathbb{N})$ is a CONS in K and if $X \in L^2_{L^2(H,K)}(\Omega, \mathcal{P}, \lambda)$,

then X is represented by

$$(1.4.3) \quad X_t = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{m,n,t} h_m \otimes k_n, \quad \text{with} \quad \|X\|^2 = \sum_m \sum_n \int_0^T E(b_{m,n,t}^2) dt$$

where $b_{m,n,t} = (X_t h_m, k_n)_K$ is a predictable process, and

$$(1.4.4) \quad \int_0^T X_t dW_t = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(\int_0^T b_{m,n,t} dW_{m,t} \right) k_n$$

where the series converges in $L_K^2(\Omega, \mathcal{A}, \mathbb{P})$.

Let X be a nonrandom function in $L_H^2(\Omega', \mathcal{P}, \lambda)$. In this case $X \in L_H^2([0, T], \mathcal{B}_{[0, T]}, dt)$, where $\mathcal{B}_{[0, T]}$ is the Borel σ -field of $[0, T]$. Then $X \rightarrow \int_0^T X_t dW_t$ is an isometry of $L_H^2([0, T], \mathcal{B}_{[0, T]}, dt)$ into $L^2(\Omega, \mathcal{A}, \mathbb{P})$ and hence, it characterizes an H -valued white noise on $[0, T]$. The passage from W to this isometry can be used in the finitely additive white noise version of the filtering problem, (cf. [10]), corresponding to an observation noise which is an H -valued Brownian motion W .

2. STOCHASTIC INTEGRATION WITH RESPECT TO NUCLEAR SPACE VALUED SQUARE INTEGRABLE MARTINGALES.

2.1 PRELIMINARIES

The topological vector spaces considered here are over the field \mathbb{R} . Given two locally convex vector spaces in duality (E, E') , $e'(e)$ or (e', e) or, if more precision is needed, $(e', e)_{E', E}$ will represent the value of $e' \in E'$ at $e \in E$. For an absolutely convex set $A \subset E$, p_A will denote its gauge. For two locally convex vector spaces E and F , the space of continuous linear mappings of E into F is denoted by $L(E, F)$. We refer to [27] for the general properties of topological vector spaces used in this section.

Let E be a complete nuclear space. If U is an absolutely convex neighborhood of 0 in E , $E(U)$ is the completion of the normed space $(E/p_U^{-1}(0), p_U)$ and $k(U)$ the canonical mapping of E into $E(U)$. For two absolutely convex neighbor-

hoods of 0, U and V in E such that $U \subset V$, the canonical mapping of $E(U)$ into $E(V)$ is denoted by $k(V, U)$ and satisfies the relation : $k(V, U) \circ k(U) = k(V)$. Since E is nuclear there exists a neighborhood base $\mathcal{U}_h(E)$ such that $\forall U \in \mathcal{U}_h(E)$, $E(U)$ is a separable Hilbert space and for all $U, V \in \mathcal{U}_h(E)$ such that $U \subset V$ the canonical mappings $k(U)$ and $k(V, U)$ are nuclear operators.

If B is a non empty closed, bounded and absolutely convex subset of E , then $E[B]$ denotes the Banach subspace of E generated by B and equipped with the norm p_B . The canonical injection of $E[B]$ into E is denoted by $i(B)$. For two bounded and absolutely convex closed subsets A and B of E such that $A \subset B$ the canonical injection of $E[A]$ into $E[B]$ is denoted by $i(B, A)$.

In this section F represents a nuclear space which is separable and complete. Its strong topological dual F' is also supposed to be complete and nuclear.

The fact that F and F' are complete nuclear spaces implies their reflexivity.

For $U \in \mathcal{U}_h(F)$, U° denotes its polar and $F'[U^\circ]$ is shown to be isometric to $F(U)'$, the topological dual of $F(U)$.

All random variables and processes considered in this section are supposed to be defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$, equipped with the filtration \mathcal{F} , as in Section 1.

A mapping $X : \Omega \rightarrow F'$ is called a weakly measurable process if for all $\phi \in F$ and all $t \in \mathbb{R}_+$, $X_t(\phi)$ is a real random variable.

We refer to [29] for an introduction to nuclear space valued martingales, and only present here, in a slightly modified form, the square integrable martingales and some of their properties. We first start with their weak characterization.

DEFINITION 2.1.1. A weakly measurable F' -valued process M is called a square integrable martingale if for all $\phi \in F$, $M(\phi) := ((M_t(\omega), \phi)_{F', F} ; (t, \omega) \in \Omega')$,

has a modification in $M^2(\mathbb{R})$. Similarly, M is said to be a continuous square integrable martingale if for all $\phi \in F$, $M(\phi)$ has a modification in $M^2_C(\mathbb{R})$.

An application of the Minlos Theorem, as presented in [28], provides the following result, for which we sketch a direct short proof.

THEOREM 2.1.2. Let \tilde{M} be a continuous linear mapping of F into $M^2(\mathbb{R})$ (resp. $M^2_C(\mathbb{R})$). Then there is an F' -valued (resp. continuous) square integrable martingale M such that, for all $\phi \in F$, $\tilde{M}(\phi)$ is a modification of $M(\phi)$.

Proof : Since F is nuclear, \tilde{M} is a nuclear mapping. Therefore, it has the following representation.

$$(2.1.1) \quad \tilde{M}(\phi) = \sum_{i=0}^{\infty} \lambda_i S_i(\phi) m_i, \quad \phi \in F$$

where $(\lambda_i) \in l^1$, $(S_i) \subset F'$ is equicontinuous and $(m_i) \subset M^2(\mathbb{R})$ (resp. $M^2_C(\mathbb{R})$) is bounded. For $\phi \in F$ and $m < n$ we note the following inequalities.

$$\begin{aligned} (2.1.2) \quad E\left(\left|\sum_{i=m}^n \lambda_i m_{i,t}(S_i, \phi)\right|\right) &\leq \sum_{i=m}^n |\lambda_i| E(\sup_t |m_{i,t}|) |(S_i, \phi)| \\ &\leq \left[\sum_{i=m}^n |\lambda_i| E(\sup_t |m_{i,t}|)\right] \sup_i |(S_i, \phi)| \\ &\leq \left(\sum_{i=m}^n |\lambda_i|\right) \sup_i E(\sup_t |m_{i,t}|) \sup_i |(S_i, \phi)| \\ &\leq 2\left(\sum_{i=m}^n |\lambda_i|\right) \sup_i [E(m_{i,\infty})^2]^{1/2} \sup_i |(S_i, \phi)| \end{aligned}$$

$$(2.1.3) \quad E\left(\sum_{i=m}^n \lambda_i m_{i,t}(S_i, \phi)\right)^2 \leq \left(\sum_{i=m}^n |\lambda_i|\right)^2 \sup_i E(m_{i,\infty})^2 \sup_i |(S_i, \phi)|^2$$

Since (S_i) also is weakly bounded the last members of these inequalities are finite. We then deduce from (2.1.2) that the set

$A = \{\omega \in \Omega; \sum_{i=0}^{\infty} |\lambda_i| \sup_t |m_{i,t}(\omega)| < \infty\}$ has probability 1.

Let us put

$$M_{n,t}(\omega) = \sum_{i=0}^n \lambda_i m_{i,t}(\omega) S_i, \quad \tilde{M}(\phi)_{n,t} = \sum_{i=0}^n \lambda_i m_{i,t}(S_i, \phi)$$

Inequalities (2.1.2) show that, for fixed $\omega \in A$, $(M_{n,t}(\omega), \phi)_{F', F} = \tilde{M}(\phi)_{n,t}(\omega)$ converges uniformly in t . Let $M_t^\phi(\omega)$ be its limit. The mapping of F into \mathbb{R} defined by $\phi \rightarrow M_t^\phi(\omega)$ for $(t, \omega) \in \mathbb{R}_+ \times A$ is linear and continuous. Therefore, it defines an element $M_t(\omega)$ in F' that we can represent by

$$(2.1.4) \quad M_t(\omega) = \sum_{i=0}^{\infty} \lambda_i m_{i,t}(\omega) S_i, \quad (t, \omega) \in \mathbb{R}_+ \times A.$$

On the other hand, the inequality (2.1.3) shows that, for all $t \in [0, \infty]$, $\tilde{M}(\phi)_{n,t}$ converges to $\tilde{M}(\phi)_t$ in $L^2(\Omega, \mathcal{A}, \mathbb{P})$. Therefore, we have for all $t \in \mathbb{R}_+$ and $\phi \in F$

$$\tilde{M}(\phi)_t(\omega) = M_t^\phi(\omega) = (M_t(\omega), \phi)_{F', F} \quad \text{a.s.}$$

From this the conclusion ensues. \square

Next is the converse of the above Theorem.

THEOREM 2.1.3. Let M be an F' -valued (resp. continuous) square integrable martingale. There then exists a continuous linear mapping \tilde{M} of F into $M^2(\mathbb{R})$ (resp. $M_C^2(\mathbb{R})$) such that for all $\phi \in F$, $M(\phi)$ is a modification of $\tilde{M}(\phi)$.

Proof : Let \tilde{M}^ϕ be the modification of $M(\phi)$ in $M^2(\mathbb{R})$ (resp. $M_C^2(\mathbb{R})$). Then by an application of the closed graph theorem the linear mapping $\tilde{M} : \phi \rightarrow \tilde{M}^\phi$ of F into $M^2(\mathbb{R})$ (resp. $M_C^2(\mathbb{R})$) is shown to be continuous. The conclusion follows from the above theorem. \square

PROPOSITION 2.1.4. Consider the following projective system of (resp. continuous) square integrable Hilbert space valued martingales.

$$(2.1.5) \quad \{M^U \in M^2(F'(U)); U \in \mathcal{U}_h(F')\}$$

$$(\text{resp. } \{M^U \in M^2_c(F'(U)); U \in \mathcal{U}_h(F')\})$$

with $M^V = k(V, U)M^U$ for $U \subset V$, (i.e. $M^V_t(\omega) = k(V, U)M^U_t(\omega)$) up to an evanescent set, which is equivalent to saying that $M^V_\infty(\omega) = k(V, U)M^U_\infty(\omega)$ a.s. There is then an F' -valued (resp. continuous) square integrable martingale M such that $\forall U \in \mathcal{U}_h(F')$, $k(U)M$ is a modification of M^U .

We say that M is the projective limit of $\{M^U; U \in \mathcal{U}_h(F')\}$.

Proof : In order to simplify the notations, we give the proof for the projective system of not necessarily continuous martingales.

Let $\tilde{M} : F \rightarrow M^2(\mathbb{R})$ be defined as follows :

$$(2.1.6) \quad \tilde{M}(\phi) := (M^U, \phi)_{F'(U), F[U^\circ]}, U \in \mathcal{U}_h(F'), \phi \in F[U^\circ].$$

For U and V in $\mathcal{U}_h(F')$ such that $U \subset V$ and $\phi \in F[V^\circ] \subset F[U^\circ]$

$$\tilde{M}(\phi) = (M^U, \phi)_{F'(U), F[U^\circ]} = (k(V, U)M^U, \phi)_{F'(V), F[V^\circ]}.$$

Therefore, $\tilde{M}(\phi)$ defined by (2.1.6) does not depend on U . It is obvious that \tilde{M} is linear. Let $(\phi_n; n \in \mathbb{N}) \subset F$ converge to some $\phi \in F$. Then (ϕ_n) is bounded and there is a neighborhood $U \in \mathcal{U}_h(F')$ such that $\{\phi, (\phi_n)\} \subset U^\circ$. Since $(M^U, \phi_n)_{F'(U), F[U^\circ]} = \tilde{M}(\phi_n)$ converges in $M^2(\mathbb{R})$ to $(M^U, \phi)_{F'(U), F[U^\circ]} = \tilde{M}(\phi)$, we see that \tilde{M} is a sequentially continuous linear mapping of F into $M^2(\mathbb{R})$. F being bornological, \tilde{M} is continuous. We then conclude by Theorem 2.1.2. \square

COROLLARY 2.1.5. A weakly measurable F' -valued process M is a (resp. continuous) square integrable martingale iff, for all $U \in \mathcal{U}_h(F')$, $k(U)M$ (defined by $(k(U)M)_t(\omega) := k(U)M_t(\omega)$) has a modification in $M^2(F'(U))$ (resp. $M_C^2(F'(U))$).

We end the paragraph with the following useful remark.

REMARK 2.1.6. Theorem 2.1.3 and Representation (2.1.4) show that given an F' -valued (resp. continuous) square integrable martingale M' there is another one, say M , having the representation (2.1.4.) such that for each t , $M'_t(\phi) = M_t(\phi)$ a.s. Since F is supposed to be separable, we have for each t , $M_t = M'_t$ a.s., i.e. M is a modification of M' .

On the other hand the set (S_i) in (2.1.4), being equicontinuous, is also bounded. Therefore there is a neighborhood $G \in \mathcal{U}_h(F)$ such that $(S_i) \subset G^\circ$. Consequently, almost all trajectories of M are in $F'[G^\circ]$. Considered as an $F'[G^\circ]$ -valued square integrable martingale, M is cadlag (resp. continuous). Since the injection $i[G^\circ]$ of $F'[G^\circ]$ into F' is continuous we see that M is a strongly cadlag (resp. continuous) F' -valued integrable martingale. This fact allows the definition of an F' -valued (resp. continuous) square integrable martingale as a strongly cadlag (resp. continuous) one. In the sequel all the F' -valued square integrable (resp. continuous) martingales will be supposed to be strongly cadlag (resp. continuous) and $M^2(F, F')$ (resp. $M_C^2(F, F')$) will denote the space of all F' -valued square integrable (resp. continuous) martingales for the duality (F, F') .

Let $L_{F'}^2(\Omega, \mathcal{A}, \mathbb{P})$ be the space of all \mathbb{P} -equivalence classes of F' -valued weakly measurable random variables X such that for all $\phi \in F$, $X(\phi) := (X, \phi)_{F', F} \in L_R^2(\Omega, \mathcal{A}, \mathbb{P})$ which is equivalent to saying that for all $U \in \mathcal{U}_h(F')$, $k(U)X \in L_{F'(U)}^2(\Omega, \mathcal{A}, \mathbb{P})$. A locally convex topology is defined on $L_{F'}^2(\Omega, \mathcal{A}, \mathbb{P})$

by the seminorms:

$$(2.1.7) \quad r_U(X) := [E(p_U^2(X))]^{1/2}, \quad U \in \mathcal{U}_h(F').$$

For $M \in M^2((F, F'))$, representation (2.1.4) holds. Therefore,

$$(2.1.8) \quad M_\infty = \sum_{i=0}^{\infty} \lambda_i m_{i,\infty} S_i = \lim_{t \rightarrow \infty} M_t \quad \text{a.s.}$$

where the limit is the strong limit in F' , and

$$(2.1.9) \quad M_t = E(M_\infty / F_t) \quad \text{a.s.}$$

It is easy to show that $M_\infty \in L_{F'}^2(\Omega, \mathcal{A}, P)$ and that M_∞ also is the limit of M_t in $L_{F'}^2(\Omega, \mathcal{A}, P)$. Consequently, with the topology defined in [29], we have $M^2(F, F') = L_{F'}^2(\Omega, \mathcal{A}, P)$.

2.2. THE INCREASING PROCESS OF A SQUARE INTEGRABLE MARTINGALE AND ITS INTEGRAL REPRESENTATION

In what follows M will represent a given martingale in $M^2(F, F')$ and $\mathcal{U}_h(F, M)$ the set of all neighborhoods U in $\mathcal{U}_h(F)$ such that M is the injection of an $F'[U^\circ]$ -valued martingale according to Remark 2.1.6.

PROPOSITION 2.2.1. There is a process $\langle M \rangle$, unique up to an evanescent set, with values in the set of symmetric, nonnegative nuclear operators in $L(F, F')$, cadlag in the bounded convergence topology of $L(F, F')$ and such that

$$(2.2.1) \quad (\langle M \rangle \phi, \psi)_{F', F} = \langle M(\phi), M(\psi) \rangle \quad \text{for all } \phi, \psi \in F \text{ except on an evanescent set.}$$

Proof : Let $U \in \mathcal{U}_h(F, M)$ and let $\langle M \rangle^U$ be the increasing process of M , considered as an $F'[U^\circ]$ -valued square integrable martingale. $\langle M \rangle^U$ has its values in $F'[U^\circ] \hat{\otimes}_1 F'[U^\circ] = L^1(F(U), F'[U^\circ])$ and it is cadlag in this space.

Let us put

$$(2.2.2) \quad \langle M \rangle^U = i(U^\circ) \circ \langle M \rangle^U \circ k(U)$$

$\langle M \rangle^U$ is also cadlag in the uniform norm topology of $L(F(U), F'[U^\circ])$. Since $i(U^\circ)$ and $k(U)$ are continuous operators, $\langle M \rangle$ is cadlag in the bounded convergence topology of $L(F, F')$. The properties of $\langle M \rangle$ other than the uniqueness are also trivial consequences of the definition (2.2.2). Suppose now there is another process A such that for all $\phi, \psi \in F$ we have $(A\phi, \psi) = (\langle M, \phi \rangle, \langle M, \psi \rangle)$ except on an evanescent set. Therefore $((\langle M \rangle - A)(\phi), \psi) = 0$ up to an evanescent set that could depend on ϕ and ψ . Since F is separable we have $((\langle M \rangle - A)(\phi), \psi) = 0$ for all ϕ, ψ in F except on an evanescent set. Therefore $\langle M \rangle$ is unique up to an evanescent set. \square

From now on $\langle M \rangle^U$ will denote, as in the preceding proof, the increasing process of M , with values in $F'[U^\circ] \hat{\otimes}_1 F[U^\circ] = L^1(F(U), F[U^\circ])$.

The uniqueness of $\langle M \rangle$ implies that its definition does not depend on the chosen neighborhood U . This fact can also be seen as follows. If $U, V \in \mathcal{U}_h(F, M)$ and if $U \subset V$, we have

$$(2.2.3) \quad \langle M \rangle^U = i(U^\circ, V^\circ) \circ \langle M \rangle^V \circ k(V, U).$$

From this we get

$$\begin{aligned} i(U^\circ) \circ \langle M \rangle^U \circ k(U) &= i(U^\circ) \circ i(U^\circ, V^\circ) \circ \langle M \rangle^V \circ k(V, U) \circ k(U) \\ &= i(V^\circ) \circ \langle M \rangle^V \circ k(V). \end{aligned}$$

Since for two arbitrary neighborhoods U and V in $\mathcal{U}_h(F, M)$ there is a third one, say W , in $\mathcal{U}_h(F, M)$, contained in $U \cap V$, (2.2.3) implies:

$$\begin{aligned} (2.2.4) \quad i(U^\circ) \circ \langle M \rangle^U \circ k(U) &= i(W^\circ) \circ \langle M \rangle^W \circ k(W) \\ &= i(V^\circ) \circ \langle M \rangle^V \circ k(V) \end{aligned}$$

Therefore, $\langle M \rangle$ is independent of the chosen neighborhood U .

DEFINITION 2.2.2. The process $\langle M \rangle$ is called the increasing process of M .

For $U \in \mathcal{U}_h(F, M)$, we denote by $\{M\}_t^U$ the increasing process defined by $\{M\}_t^U = ||\langle M \rangle_t^U||_1$ and by Q^U the predictable process with values in the set of symmetric non-negative elements of $L^1(F(U), F'[U^\circ])$ characterized by $d\langle M \rangle_t^U = Q_t^U d\{M\}_t^U$. We denote by λ^U the measure on P defined by $dP d\{M\}_t^U$. We shall prove that all these measures are equivalent to each other. For this purpose we first need to prove the following lemma.

LEMMA 2.2.3. For all $A \in P$ and all $\phi \in F$ the quantity

$$\int_A (Q^U(k(U)\phi), k(U)\phi) d\lambda^U$$

is independent of $U \in \mathcal{U}_h(F, M)$.

Proof: Let us choose $U, V \in \mathcal{U}_h(F, M)$, $A =]s, t] \times B$ with $s < t$ and $B \in \mathcal{F}_s$.

Then we have

$$\begin{aligned} \int_A (Q^U(k(U)\phi), k(U)\phi) d\lambda^U &= E[1_B \cdot (\langle M \rangle_t^U - \langle M \rangle_s^U)(k(U)\phi, k(U)\phi)] \\ &= E[1_B \cdot ((\langle M \rangle_t - \langle M \rangle_s)\phi, \phi)] \\ &= \int_A (Q^V(k(V)\phi), k(V)\phi) d\lambda^V \end{aligned}$$

If $A = \{0\} \times B$ with $B \in \mathcal{F}_0$, similarly, we have

$$\int_A (Q^U(k(U)\phi), k(U)\phi) d\lambda^U = \int_A (Q^V(k(V)\phi), k(V)\phi) d\lambda^V$$

This shows that the measures defined by $A \rightarrow \int_A (Q^U(k(U)\phi), k(U)\phi) d\lambda^U$ and $A \rightarrow \int_A (Q^V(k(V)\phi), k(V)\phi) d\lambda^V$, coincide on the set of predictable rectangles. Therefore they coincide on P . \square

THEOREM 2.2.4. The measures λ^U , $U \in \mathcal{U}_h(F, M)$, are all equivalent to each other.

Proof: Let us choose $U, V \in \mathcal{U}_h(F, M)$ and suppose that $\lambda^U(A) = 0$ for some $A \in P$. Then we can write

$$\forall \phi \in F, \int_A ((Q^U \circ k(U))\phi, k(U)\phi) d\lambda^U = 0$$

But according to Lemma 2.2.3

$$\forall \phi \in F, \int_A ((Q^V \circ k(V))\phi, k(V)\phi) d\lambda^V = 0$$

Let $(e_j; j \in \mathbb{N})$ be a total family in F . Then $(k(V)e_j; j \in \mathbb{N})$ is a total family in $F(V)$. Let us denote by $(k(V)e_{j_n}; n = 0, 1, 2, \dots)$ the CONS in $F(V)$ obtained from $(k(V)e_j; j \in \mathbb{N})$ by the Gram-Schmidt orthogonalization method. (A finite dimensional $F(V)$ is not excluded!). The last equality implies

$$\sum_n \int_A ((Q^V \circ k(V))e_{j_n}, k(V)e_{j_n}) d\lambda^V = 0.$$

Consequently $\int_A \text{Tr } Q^V d\lambda^V = 0$. But $\text{Tr } Q^V = 1$, λ^V -a.e. Its integral on A can vanish only if $\lambda^V(A) = 0$. Therefore, λ^V is absolutely continuous with respect to λ^U . Since U and V are arbitrary, λ^U is also absolutely continuous with respect to λ^V . \square

From now on we choose once and for all a neighborhood $G \in \mathcal{U}_h(F, M)$ and use $F[G^*]$ ($\cong F(G)$) as the reference Hilbert space such that M can be considered as a $F[G^*]$ -valued process. Then we denote $\{M\}^G$ simply by $\{M\}$ and λ^G by λ .

PROPOSITION 2.2.5. Let the $L(F, F')$ -valued process Q be defined by

$$(2.2.5) \quad Q := i(G^\circ) \circ Q^G \circ k(G)$$

Then for λ -almost all $(t, \omega) \in \Omega'$, $Q_t(\omega)$ is a symmetric, nonnegative nuclear operator and for all $\phi, \psi \in F$, $(Q\phi, \psi)$ is a real predictable process which is integrable with respect to λ . Moreover,

$$(2.2.6) \quad \left\{ \begin{array}{l} \int_{]s, t] \times B} (Q\phi, \psi) d\lambda = E[1_B \cdot ((\langle M \rangle_t - \langle M \rangle_s)(\phi), \psi)], \quad s < t, B \in F_s \\ \int_{\{0\} \times B} (Q\phi, \psi) d\lambda = E[1_B \cdot (\langle M \rangle_0(\phi), \psi)], \quad B \in F_0 \end{array} \right.$$

Q depends on the chosen neighborhood G , but is unique λ -a.e. once G is chosen, and we have

$$(2.2.7) \quad d\langle M \rangle = Q \, d\langle M \rangle$$

Proof: All the mentioned properties of Q are immediate consequences of the definition (2.2.5). The uniqueness is the consequence of (2.2.6). We just prove these equalities.

Let $\phi, \psi \in F$, $s < t$ and $B \in F_s$. According to (2.2.2) we have

$$\begin{aligned} E[1_B \cdot ((\langle M \rangle_t - \langle M \rangle_s)(\phi), \psi)] &= E[1_B \cdot ((\langle M \rangle_t^G - \langle M \rangle_s^G) \circ k(G)\phi, k(G)\psi)] = \\ &= \int_{]s, t] \times B} ((Q^G \circ k(G))\phi, k(G)\psi) d\lambda \end{aligned}$$

and similarly, for $B \in F_0$

$$E[1_B \cdot (\langle M \rangle_0(\phi), \psi)] = \int_{\{0\} \times B} ((Q^G \circ k(G))(\phi), k(G)\psi) d\lambda.$$

These equalities together with the definition (2.2.5) of Q imply the equalities

(2.2.6). \square

2.3. STOCHASTIC INTEGRATION

The construction of the stochastic integral with respect to M is very similar to the one presented in Section 1 for Hilbert space valued square integrable martingales with closer attention paid to measurability problems.

We suppose again that the neighborhood $G \in \mathcal{U}_h(F, M)$ is fixed once and for all, so that M can be considered as an $F'[G^\circ]$ -valued square integrable martingale. λ and Q are those defined in the preceding paragraph. We identify $F(G)$ with $F'[G^\circ]$ and denote it by H . Operations on Q are only valid λ -a.e., and in order to simplify the notations we will not always mention it.

Let D^G be an $L^2(H, H)$ -valued predictable process such that $D^G \circ D^{G*} = Q^G$ λ -a.e. and let us put

$$(2.3.1) \quad D := i(G^\circ) \circ D^G$$

Then we have the factorization $D \circ D^* = Q$. We note that $D_t^*(\omega)$ is a nuclear operator from F into H .

$L^{*2}(D_t(\omega), F)$ will be the vector space of linear operators f of F' into R such that $Rg \ D_t(\omega) \subset \text{Dom } f$ and that $f \circ D_t(\omega)$ is continuous on H . Equipped with the scalar product $(f, g) := ((f \circ D_t(\omega))^*, (g \circ D_t(\omega))^*)_H$, this space becomes a Hilbert space. Here $f \circ D_t(\omega)$ represents the continuous functional $f \circ D_t(\omega)$ on H . Let us denote (here again!) by $\tilde{H}_t(\omega)$ the Hilbert subspace of $L^{*2}(D_t(\omega), F)$ generated by F whose elements are considered as continuous linear functionals on F' . For all $f, g \in F$ we have $(f, g)_{\tilde{H}_t(\omega)} = (D_t^*(\omega)f, D_t^*(\omega)g)_H$. As

in Section 1 we shall define a predictable field of Hilbert spaces in $\prod_{\Omega'} \tilde{H}_t(\omega)$ and construct the stochastic integral for square-integrable fields. But we need beforehand to consider some measurability problems.

We say that an F -valued process X is weakly predictable if for all $f \in F'$, $(f, X)_{F', F}$ is a real predictable process. If X is weakly predictable then for any $h \in H$, $(i(G^\circ)h, X)_{F', F} = (h, k(G)X)_H$ is predictable. Therefore, $D^*X = D^{G^*} \circ k(G)X$ is also predictable, because so is D^{G^*} . Similarly, if X and Y are F -valued weakly predictable processes, then

$$\begin{aligned} (X_t(\omega), Y_t(\omega))_{\tilde{H}_t(\omega)} &= (D_t^*(\omega)X_t(\omega), D_t^*(\omega)Y_t(\omega))_H \\ &= (D_t^{G^*}(\omega)(k(G)X_t(\omega)), D_t^{G^*}(\omega)(k(G)Y_t(\omega)))_H \end{aligned}$$

defines a predictable process.

Let $e = (e_n; n \in \mathbb{N})$ be a linearly independent sequence generating F . We could choose e in such a way that $(k(G)e_n; n \in \mathbb{N})$ is a linearly independent sequence generating $F(G)$. Let us again consider the sequence $\tilde{e}_t(\omega) = (\tilde{e}_{n,t}(\omega), n \in \mathbb{N})$ obtained from e by the procedure of (1.2.2). Each \tilde{e}_n is an F -valued weakly predictable process and $k(G)\tilde{e}_n$ is an H -valued predictable one. $E(D, F)$ is the predictable field in $\prod_{\Omega} \tilde{H}_t(\omega)$ consisting of elements X such that $(X_t(\omega), \tilde{e}_{n,t}(\omega))_{\tilde{H}_t(\omega)}$ is predictable for all $n \in \mathbb{N}$. As in Section 1 we can see that $E(D, F)$ is generated by the set of all F -valued weakly predictable processes and even by the set of all F -valued constant processes.

$\tilde{\Lambda}^2(D, F)$ is the Hilbert space of all elements X of $E(D, F)$ such that

$$(2.3.2) \quad ||X||_{\tilde{\Lambda}^2}^2 := \int_{\Omega} ||X_t(\omega)||_{\tilde{H}_t(\omega)}^2 \lambda(dt, d\omega) < \infty.$$

If $X \in \tilde{\Lambda}^2(D, F)$ then it has the following representation.

$$(2.3.3) \quad X_t(\omega) = \sum_{n=0}^{\infty} a_{n,t}(\omega) \tilde{e}_{n,t}(\omega) \quad \lambda\text{-a.e.}$$

with predictable coefficients such that

$$(2.3.4) \quad \|X\|_{\tilde{\Lambda}^2}^2 = \sum_{n=0}^{\infty} \int_{\Omega} a_{n,t}^2(\omega) \lambda(dt, d\omega).$$

As in the setting of Section 1, (cf. Remark 1.2.6) we can prove that $\tilde{\Lambda}^2(D, F)$ is isometric with $\tilde{\Lambda}^2(D^G, H)$. An F -valued process X is said to be elementary if it is of the form

$$(2.3.5) \quad X := 1_{\{0\} \times B_0} \phi_0 + \sum_{i=1}^n 1_{]t_i, t_{i+1}]} \times B_i \phi_i$$

where $0 = t_0 < t_1 < \dots$, $B_i \in \mathcal{F}_{t_i}$ and $\phi_i \in F$.

The isometric image of X in $\tilde{\Lambda}^2(D^G, H)$, given by (2.3.5), is $k(G)X$. Since the class of all elementary processes of type $k(G)X$ is dense in $\tilde{\Lambda}^2(D^G, H)$ we see that F -valued elementary processes generate $\tilde{\Lambda}^2(D, F)$.

The stochastic integral of the elementary process X of (2.3.5) is the ordinary Stieltjes integral:

$$(2.3.6) \quad \int_{\mathbb{R}_+} X dM := 1_{B_0} M_0(\phi_0) + \sum_{i=0}^n 1_{B_i} [M_{t_{i+1}}(\phi_i) - M_{t_i}(\phi_i)].$$

$X \mapsto \int_{\mathbb{R}_+} X dM$ is an isometry of the set of all elementary processes into $L^2(\Omega, \mathcal{A}, \mathbb{P})$. Then for an arbitrary element of $\tilde{\Lambda}^2(D, F)$ the stochastic integral is defined by the extension of this isometry to $\tilde{\Lambda}^2(D, F)$. The square integrable real martingale defined by $\int_{0-}^t X dM$ is denoted by $X \cdot M$.

The stochastic integral can also be defined by means of the strongly orthogonal sequence $(M_n = \tilde{e}_n \cdot M; n \in \mathbb{N})$. In fact if X is given by (2.3.3) then we have

$$(2.3.7) \quad \int_{\mathbb{R}_+} X_t dM_t = \sum_{n=0}^{\infty} \int_{\mathbb{R}_+} a_{n,t} dM_{n,t}$$

where the series converges in $L^2(\Omega, \mathcal{A}, \mathbb{P})$.

For a separable Hilbert space K we construct $\tilde{\Lambda}^2(D, F, K)$ exactly in the same way as the space $\tilde{\Lambda}^2(D, H, K)$ in Section 1. If $X \in \tilde{\Lambda}^2(D, F, K)$ then it is represented by

$$(2.3.8) \quad X_t(\omega) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{m,n,t}(\omega) \tilde{e}_{m,t}(\omega) \otimes k_n$$

where $(k_n; n \in \mathbb{N})$ is a CONS in K , the coefficients are predictable and

$$(2.3.9) \quad \|X\|_{\tilde{\Lambda}^2}^2 = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \int_{\Omega} b_{m,n}^2 d\lambda < \infty$$

The stochastic integral of X is again defined by

$$(2.3.10) \quad \int_{\mathbb{R}_+} X_t dM_t = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(\int_{\mathbb{R}_+} b_{m,n,t} dM_{m,t} \right) k_n$$

where the series converges in $L_K^2(\Omega, \mathcal{A}, \mathbb{P})$. We always denote by $X \cdot M$ the K -valued square integrable martingale defined by $\int_0^t X dM$.

Before ending this paragraph we make the following observation. Spaces $\tilde{\Lambda}^2(D, F, K)$ and $\tilde{\Lambda}^2(D^G, F(G), K)$ are isometric. In fact it is easily seen that if $X \in \tilde{\Lambda}^2(D, F, K)$ is given by (2.3.8) then its isometric image X^G in $\tilde{\Lambda}^2(D^G, F(G), K)$ is represented by

$$X_t^G(\omega) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{m,n,t}(\omega) (k(G) \tilde{e}_{m,t}(\omega)) \otimes k_n.$$

Consequently the integral $\int_{\mathbb{R}_+} X^G dM$, with M considered as $F'[G^\circ]$ -valued is again given by (2.3.10) where M_m is $(k(G) \tilde{e}_m) \cdot M$ and represents the same martingale M_m defined by $\tilde{e}_m \cdot M$ when M is taken to be F' -valued.

A close look at the construction of $\tilde{\Lambda}^2(D, F)$ and the corresponding stochastic integrals shows that everything depends on the representation $d\langle M \rangle_t(\omega) = i(G^\circ) \circ Q_t^G(\omega) \circ k(G) d\tilde{M}_t^G(\omega)$. Let us suppose that $\langle M \rangle$ also has another representation $d\langle M \rangle_t(\omega) = A^* \circ Q_t^J(\omega) \circ AdZ_t(\omega)$ through a separable Hilbert space J , where A is a continuous linear mapping of F into J , Q^J a predictable process with values in the set of positive symmetric operators in $L^1(J, J)$ such that $\|Q_t^J(\omega)\|_1$ is bounded, and Z is an increasing predictable positive cadlag process such that $\sup_t E(Z_t) < \infty$. Then what we have done can be repeated with the new representation of $\langle M \rangle$ for the construction of the corresponding spaces $\tilde{H}_t(\omega)$. The new $\tilde{H}_t(\omega)$ is isometric with the one we have been considering here. We can also choose for $\tilde{H}_t(\omega)$ any abstract completion of F under the scalar product $(\phi, \psi)_{\tilde{H}_t(\omega)} = (A^* \circ Q_t^J(\omega) \circ A\phi, \psi)_{F', F}$. All the spaces $\tilde{\Lambda}^2$ corresponding to various representations of $\langle M \rangle$ as above are isometric and the stochastic integrals of isometric elements give the same random variables. We then see that the set of martingales obtained by stochastic integration with respect to M does not depend on the chosen particular representation of $\langle M \rangle$.

2.4. STOCHASTIC INTEGRAL REPRESENTATION OF MARTINGALES

We consider here a pair (E, E') of nuclear spaces in duality having exactly the same properties as the pair (F, F') we have been considering in this section.

We continue to use the same notations as in the preceding paragraph.

Since the canonical mapping $k(U)$ of E' into $E'(U)$, with $U \in \mathcal{U}_h(E')$ is nuclear, for any continuous linear mapping A from $\tilde{H}_t(\omega)$ into E' , the mapping $k(U) \circ A$ of $\tilde{H}_t(\omega)$ into $E'(U)$ is a nuclear, hence a Hilbert-Schmidt operator.

We denote by $\tilde{\Lambda}^2(D, F, E')$ the space of all "processes" A such that

- a) for λ -almost all $(t, \omega) \in \Omega'$, $A_t(\omega)$ is a continuous linear operator from $H_t(\omega)$ into E' .

- b) for all $\phi \in F$ and $\psi \in E$, $(A_t(\omega)\phi, \psi)_{E^-, E}$ defines a real predictable process, i.e. A is weakly predictable
- c) for all $U \in U_h(E^-)$, $k(U) \circ A_t(\omega)$ defines an element of $\tilde{\Lambda}^2(D, F, E^-(U))$ that we denote by $k(U) \circ A$.

For $A \in \tilde{\Lambda}^2(D, F, E^-)$ we define the seminorms $(q_U; U \in U_h(E^-))$ by

$$(2.4.1) \quad q_U(A) := \left(\int_{\Omega^-} \|k(U) \circ A_t(\omega)\|^2_{\tilde{H}_t(\omega) \otimes E^-(U)} \lambda(dt, d\omega) \right)^{1/2}.$$

With the vector space topology induced by these seminorms the space $\tilde{\Lambda}^2(D, F, E^-)$ is a locally convex space. Divided by the equivalence relation $A \sim B \iff A = B$ λ -a.e. it becomes a Hausdorff space. We again denote the quotient space by $\tilde{\Lambda}^2(D, F, E^-)$.

The following theorem shows that the elements of $\tilde{\Lambda}^2(D, F, E^-)$ can be represented bijectively as projective systems.

THEOREM 2.4.1. Let $(A^U \in \tilde{\Lambda}^2(D, F, E^-(U)); U \in U_h(E^-))$ be a projective system with respect to $k(V, U)$, i.e. $U \subset V \implies k(V, U) \circ A^U = A^V$ λ -a.e. Then there is a unique process $A \in \tilde{\Lambda}^2(D, F, E^-)$ such that $k(U) \circ A = A^U$ λ -a.e..

Proof. We consider the continuous linear mapping γ of E into $\tilde{\Lambda}^2(D, F)$ defined by $\gamma\phi = (A^U)^*\phi$ for $\phi \in E[U^\circ]$, $U \in U_h(E^-)$. We first show that

$(A^U)^*\phi \in \tilde{\Lambda}^2(D, F)$. Since $A_t^U(\omega): \tilde{H}_t(\omega) \rightarrow E^-(U)$ then $(A_t^U(\omega))^*: E[U^\circ] \rightarrow \tilde{H}_t(\omega)$.

Let ϕ be a nonzero element of $E[U^\circ]$ and let us again denote by ϕ its nonzero isometric image in $E^-(U)$. If $(k_n; n \in \mathbb{N})$ is a CONS in $E^-(U)$ such that $k_0 = \phi / \|\phi\|_{E^-(U)}$, then we can write

$$A_t^U(\omega) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n,t}(\omega) \tilde{e}_{m,t}(\omega) \otimes k_n;$$

and we have

$$(A_t^U(\omega))^* \phi = ||\phi||_{E^-(U)} \sum_{m=0}^{\infty} a_{m,0,t}(\omega) \tilde{e}_{m,t}(\omega).$$

Since $\sum_{m=0}^{\infty} \int_{\Omega^-} a_{m,0}^2 d\lambda < \infty$, we see that $(A^U)^* \phi \in \tilde{\Lambda}^2(D, F)$. For $U, V \in U_h(E^-)$ with $U \subset V$ and $\phi \in E[U^\circ]$, we have $k(V, U) \circ A^U = A^V$ λ -a.e.. Therefore $(A^U)^* \phi = (A^V)^* \phi$ λ -a.e.. This implies that the λ -equivalence class of $(A^U)^* \phi$ does not depend on the neighborhood $U \in U_h(E^-)$. Hence, $\gamma: E \rightarrow \tilde{\Lambda}^2(D, F)$ is well defined. Its linearity is obvious. Now we prove its continuity.

Let $(\phi_n, n \in \mathbb{N})$ converge to ϕ in $E[U^\circ]$ for some $U \in U_h(E^-)$. Since

$$|| (A^U)^* (\phi - \phi_n) ||_{\tilde{\Lambda}^2(D, F)} \leq ||A^U||_{\tilde{\Lambda}^2(D, F, E^-(U))} ||\phi - \phi_n||_{E[U^\circ]},$$

$\gamma \circ i(U^\circ)\phi_n = (A^U)^* \phi_n$ converges to $\gamma \circ i(U^\circ)\phi = (A^U)^* \phi$ in $\tilde{\Lambda}^2(D, F)$.

As any convergent sequence in E can be taken to belong to some $E[U^\circ]$,

$U \in U_h(E^-)$, we see that $\gamma: E \rightarrow \tilde{\Lambda}^2(D, F)$ is sequentially continuous.

E being a bornological nuclear space, γ is continuous and hence nuclear.

Therefore γ has a representation of the following type.

$$(2.4.2) \quad (\gamma\phi)_t(\omega) = \sum_{n=0}^{\infty} a_n S_n(\phi) \chi_{n,t}(\omega) \quad \lambda\text{-a.e.} \quad \phi \in E$$

where $(a_n) \in l^1$, (S_n) is equicontinuous in E' and (χ_n) is bounded in $\tilde{\Lambda}^2(D, F)$.

Since $(\chi_n; n \in \mathbb{N})$ is bounded in $\tilde{\Lambda}^2(D, F)$ we have

$$\begin{aligned} & \int_{\Omega^-} \sum_{n=0}^{\infty} |a_n| ||\chi_{n,t}(\omega)||_{H_t(\omega)}^2 \lambda(dt, d\omega) \\ &= \sum_{n=0}^{\infty} |a_n| \int_{\Omega^-} ||\chi_{n,t}(\omega)||_{H_t(\omega)}^2 \lambda(dt, d\omega) \end{aligned}$$

$$\leq \left(\sum_{n=0}^{\infty} |a_n| \right) [\lambda(\Omega^-)]^{1/2} \left(\sup_n \|x_n\|_{\Lambda} \right) < \infty$$

Therefore, $\sum_{n=0}^{\infty} |a_n| \|x_{n,t}(\omega)\|_{H_t(\omega)} < \infty$ λ -a.e. and this implies that

$$\left\| \sum_{n=0}^{\infty} a_n x_{n,t}(\omega) S_n(\phi) \right\|_{H_t(\omega)} \leq \left(\sum_{n=0}^{\infty} |a_n| \|x_{n,t}(\omega)\|_{H_t(\omega)} \right) \sup_n |S_n(\phi)| < \infty$$

because $(S_n; n \in \mathbb{N})$ is also bounded in E' .

If $(\phi_j; j \in \mathbb{N})$ converges to ϕ in E , then by the above inequality and the equicontinuity of $(S_n; n \in \mathbb{N})$ we have $\lim_{j \rightarrow \infty} \left\| \sum_{n=0}^{\infty} a_n x_{n,t}(\omega) S_n(\phi_j - \phi) \right\|_{H_t(\omega)} = 0$.

Therefore the linear mapping $C_t(\omega): E \rightarrow \tilde{H}_t(\omega)$ defined by

$$(2.4.3) \quad C_t(\omega)\phi := \sum_{n=0}^{\infty} a_n S_n(\phi) x_{n,t}(\omega)$$

is continuous and hence nuclear.

Now, we put $A_t(\omega) := C_t(\omega)^*$. Then $A_t(\omega)$ is a continuous mapping of $\tilde{H}_t(\omega)$ into E' . For $\phi \in F$ and $\psi \in E$, we have

$$(A_t(\omega)\phi, \psi)_{E', E} = (\phi, C_t(\omega)\psi)_{\tilde{H}_t(\omega)} = (\phi, (\gamma\psi)_t(\omega))_{\tilde{H}_t(\omega)} \quad \lambda\text{-a.e.}$$

Therefore, A is a weakly predictable process.

Finally, for $U \in \mathcal{U}_h(E')$, $\phi \in E[U^\circ]$ and $h \in \tilde{H}_t(\omega)$, we have

$$\begin{aligned} (k(U) \circ A_t(\omega)h, \phi)_{E'(U), E[U^\circ]} &= (h, A_t^*(\omega) \circ i(U^\circ)\phi)_{\tilde{H}_t(\omega)} \\ &= (h, A_t^*(\omega)\phi)_{\tilde{H}_t(\omega)} = (h, C_t(\omega)\phi)_{\tilde{H}_t(\omega)} = (h, (A_t^U(\omega))^*\phi)_{\tilde{H}_t(\omega)}. \end{aligned}$$

Consequently, $k(U) \circ A_t(\omega) = A_t^U(\omega)$ a.e..

Suppose now that there is another element \tilde{A} of $\tilde{\Lambda}^2(D, F, E')$ such that for each $U \in \mathcal{U}_h(E')$, $k(U) \circ \tilde{A} = A^U$ λ -a.e. We then have $q_U(A - \tilde{A}) = 0$ for $U \in \mathcal{U}_h(E')$. Therefore, $A = \tilde{A}$ λ -a.e., i.e. A is unique in $\tilde{\Lambda}^2(D, F, E')$. \square

REMARK 2.4.2. The above proof shows that if A is an element of $\tilde{\Lambda}^2(D, F, E')$ then it has the following representation.

$$(2.4.4) \quad A_t(\omega) = \sum_{n=0}^{\infty} a_n(X_{n,t}(\omega), \cdot) \tilde{H}_t(\omega) S_n \quad \lambda\text{-a.e.}$$

where $(a_n) \in l^1$, (S_n) is equicontinuous in E' and (X_n) is bounded in $\tilde{\Lambda}^2(D, F)$.

For a given $A \in \tilde{\Lambda}^2(D, F, E')$, we define the following projective system of square integrable martingales.

$$(2.4.5) \quad N_t^U = \int_{0-}^t k(U) \circ A_s dM_s, \quad U \in \mathcal{U}_h(E')$$

i.e. for $U, V \in \mathcal{U}_h(E')$ such that $U \subset V$, the equality $k(V, U)N^U = N^V$ holds. Therefore according to Proposition 2.1.4, there is a martingale $N \in M^2(E, E')$ such that for all $U \in \mathcal{U}_h(E')$, $k(U)N = N^U$, i.e. N is the limit of the projective system $(N^U; U \in \mathcal{U}_h(E'))$.

DEFINITION 2.4.3. Whenever $N \in M^2(E, E')$ is defined by the projective system

$((k(U) \circ A) \cdot M; U \in \mathcal{U}_h(E'))$ for some $A \in \tilde{\Lambda}^2(D, F, E')$, we define $\int_{0-}^t A_s dM_s$ by N_t and we put $N = A \cdot M$.

THEOREM 2.4.4. The mapping $A \mapsto A \cdot M$ defines an algebraic and topological isomorphism of $\tilde{\Lambda}^2(D, F, E')$ into $M^2(E, E')$.

Proof. The linearity of the stochastic integral is obvious. For $U \in \mathcal{U}_h(E')$

we have $E ||k(U) \int_{\mathbb{R}_+} A \, dM||_{E'(U)}^2 = ||k(U) \circ A||^2$ where the last norm is computed in $\tilde{\Lambda}^2(D, F, E'(U))$. This, together with the uniqueness of A , shows that $A \mapsto A.M$ is a topological isomorphism. \square

The proof of Theorem 2.4.1. suggests the following result, analogous to the one mentioned in Remark 2.1.6.

PROPOSITION 2.4.5. Let an E' -valued square integrable martingale N be given by the projective system $(N^U := A^U.M; U \in \mathcal{U}_h(E'))$, where $(A^U; U \in \mathcal{U}_h(E'))$ is the projective system of Theorem 2.4.1. Then there is a neighborhood $V \in \mathcal{U}_h(E)$ and a process $A' \in \tilde{\Lambda}^2(D, F, E'[V^\circ])$ such that $N = (i(V^\circ) \circ A').M = i(V^\circ)(A'.M)$.

Proof. Since the sequence $(S_n; n \in \mathbb{N})$ in (2.4.4) is bounded in E' , there is a neighborhood $V \in \mathcal{U}_h(E)$ such that $(S_n; n \in \mathbb{N}) \subset V^\circ$. Now, let us consider the sequence $(A'_n; n \in \mathbb{N})$ in $\tilde{\Lambda}^2(D, F, E'[V^\circ])$ defined by

$$A'_{n,t}(\omega) := \sum_{k=0}^n a_k X_{k,t}(\omega) \odot S_k \in \tilde{H}_t(\omega) \hat{\otimes}_2 E'[V^\circ].$$

We have for $m < n$ and $c = \sup_n p_{V^\circ}^2(S_n)$,

$$\begin{aligned} & \int_{\Omega^-} || \sum_{k=m}^n a_k X_{k,t}(\omega) \odot S_k ||_{\tilde{H}_t(\omega) \hat{\otimes}_2 E'[V^\circ]}^2 \lambda(dt, d\omega) \\ & \leq c \int_{\Omega^-} \left(\sum_{k=m}^n |a_k| ||X_{k,t}(\omega)||_{\tilde{H}_t(\omega)} \right)^2 \lambda(dt, d\omega) \\ & = c \int_{\Omega^-} \sum_{k=m}^n \sum_{j=m}^n |a_k| |a_j| ||X_{k,t}(\omega)|| ||X_{j,t}(\omega)|| \lambda(dt, d\omega) \\ & \leq c \left(\sum_{k=m}^n |a_k| \right)^2 \sup_k \left(\int_{\Omega^-} ||X_{k,t}(\omega)||_{\tilde{H}_t(\omega)}^2 \lambda(dt, d\omega) \right). \end{aligned}$$

We see that $(A_n^-; n \in \mathbb{N})$ is a Cauchy sequence in $\tilde{\Lambda}^2(D, F, E'[V^\circ])$ and its limit A^- can be represented by

$$A_t^-(\omega) := \sum_{n=0}^{\infty} a_n X_{n,t}(\omega) \otimes S_n \quad \lambda\text{-a.e.}$$

Let us define $A_n \in \tilde{\Lambda}^2(D, F, E^-)$ by $A_{n,t}(\omega)\phi := \sum_{k=0}^n a_k (X_{k,t}(\omega), \phi)_{\tilde{H}_t(\omega)} S_k$,

$\phi \in \tilde{H}_t(\omega)$. We can formally represent A_n by

$$A_{n,t}(\omega) = \sum_{k=0}^n a_k X_{k,t}(\omega) \otimes S_k \in \tilde{H}_t(\omega) \otimes E^-.$$

On the other hand, we deduce from (2.4.4) that for $U \in \mathcal{U}_h(E^-)$, $k(U) \circ A$ has the following representation:

$$k(U) \circ A_t(\omega) = \sum_{n=0}^{\infty} a_n X_{n,t}(\omega) \otimes (k(U)S_n) \in \tilde{H}_t(\omega) \hat{\otimes}_2 E^-(U)$$

From this we can also deduce that A_n converges to A in $\tilde{\Lambda}^2(D, F, E^-)$.

We observe that the mappings $i(V^\circ): B \rightarrow i(V^\circ) \circ B$ defined by $(i(V^\circ) \circ B)_t(\omega) = i(V^\circ) \circ B_t(\omega)$ from $\tilde{\Lambda}^2(D, F, E'[V^\circ])$ into $\tilde{\Lambda}^2(D, F, E^-)$ and $i(V^\circ): N \rightarrow i(V^\circ) \circ N$ defined by $(i(V^\circ) \circ N)_t(\omega) = i(V^\circ)N_t(\omega)$ from $M^2(E'[V^\circ])$ into $M^2(E, E^-)$ are continuous. We then see that $A_n = i(V^\circ) \circ A_n^-$ converges to $A = i(V^\circ) \circ A^-$ as $n \rightarrow \infty$. Therefore, $A_n.M = (i(V^\circ) \circ A_n^-).M = i(V^\circ)(A_n^-.M)$ converges to $A.M = (i(V^\circ) \circ A^-).M = i(V^\circ)(A^-.M)$. \square

In what follows we shall give the extension of Theorem 1.3.5 for the representation of nuclear space valued square integrable martingales.

We first consider the following trivial extension.

THEOREM 2.4.6. Let N be a square integrable martingale with values in the separable real Hilbert space K . Then there is a process $X \in \tilde{\Lambda}^2(D, F, K)$

such that

$$(2.4.6) \quad N = X.M + N^\perp$$

where N^\perp is a K -valued square integrable martingale orthogonal to M . The above representation is unique in the sense that X is unique up to λ -equivalence and N^\perp is unique up to an evanescent set.

The orthogonality of N^\perp with M is expressed as follows: $\forall \phi \in F, \forall k \in K,$
 $\langle (M, \phi)_{F, F}, (N^\perp, k)_K \rangle = 0$ up to an evanescent set.

Proof. By Theorem 1.3.5 we can uniquely represent N by $X^G.M + N^\perp$ where $X^G \in \tilde{\Lambda}^2(D^G, F(G) \cong F[G^\circ], K)$ and M is considered as $F[G^\circ]$ -valued. But the martingale $X^G.M$ is also represented by $X.M$ with $X \in \tilde{\Lambda}^2(D, F, K)$. The orthogonality of N^\perp to M is obvious. \square

THEOREM 2.4.7. Let N be an E' -valued square integrable martingale ; then N has the following representation

$$(2.4.7) \quad N = X.M + N^\perp$$

where $X \in \tilde{\Lambda}^2(D, F, E')$ and $N^\perp \in M^2(E, E')$ is orthogonal to M . This decomposition is unique in the sense that X is unique up to λ -equivalence and N^\perp is unique up to an evanescent set.

Here again the orthogonality of N^\perp with M is expressed by $\forall \phi \in F, \forall \psi \in E$ we have $\langle (M, \phi), (N^\perp, \psi) \rangle = 0$ up to an evanescent set.

Proof. Let us consider the projective system $(k(U)N, U \in \mathcal{U}_h(E'))$ whose limit is N . Then according to Theorem 1.3.5,

$$(2.4.8) \quad k(U)N = X^U.M + N^{U\perp}$$

where $X^U \in \tilde{\Lambda}^2(D, F, E'(U))$ and $N^{U\perp} \in M^2(E'(U))$ is orthogonal to M . For $U, V \in \mathcal{U}_h(E')$ such that $U \subset V$, we have $k(V, U)(X^U.M + N^{U\perp}) = k(V, U)X^U.M + k(V, U)N^{U\perp} = X^V.M + N^{V\perp}$ and $\forall \phi \in F, \forall h \in E'(V)$

$$\langle (M, \phi)_{F', F}, (k(V, U)N^{U\perp}, h)_{E'(V)} \rangle = \langle (M, \phi)_{F', F}, (N^{U\perp}, k^*(V, U)h)_{E'(U)} \rangle = 0.$$

Hence, by the uniqueness of the orthogonal decomposition, $k(V, U)(X^U.M) = X^V.M$ and $k(V, U)N^{U\perp} = N^{V\perp}$. Therefore $(X^U.M; U \in \mathcal{U}_h(E'))$ and $(N^{U\perp}; U \in \mathcal{U}_h(E'))$ are projective systems. According to Theorem 2.4.1, there is an element X of $\tilde{\Lambda}^2(D, F, E')$ unique up to λ -equivalence such that the projective limit of $(X^U.M; U \in \mathcal{U}_h(E'))$ is represented by $X.M$. If N^\perp denotes the projective limit of $(N^{U\perp}; U \in \mathcal{U}_h(E'))$ then the decomposition (2.4.7) is obtained. The uniqueness of the decomposition is a consequence of the uniqueness of the decomposition (2.4.8). \square

As in the case of Hilbert space valued martingales we can give the integral representation of M by itself.

PROPOSITION 2.4.8. We have

$$(2.4.9) \quad M_t = \int_{0-}^t D_s \circ I_s \, dM_s$$

where $I_t(\omega)$ is the isometry of $\tilde{H}_t(\omega)$ into $H = F(G) \cong F[G^\circ]$.

Proof. Immediate consequence of Proposition 1.3.6. \square

Before ending this Section, we would like to point out some topological facts on locally convex tensor products that could provide a better understanding of stochastic integration. What we recall here below on locally convex tensor products was developed in [11] as an extension of Chevet and Saphar's works [2] and [26].

Let E be an arbitrary locally convex space and $U(E)$ be a base of absolutely convex closed neighborhoods of 0 in E . For a positive number k , the conjugate number k' is the one determined by $\frac{1}{k} + \frac{1}{k'} = 1$. If $E \hat{\otimes}_T F$ is a locally convex tensor product of two locally convex spaces E and F , then $E \hat{\otimes}_T F$ denotes its completion. For an index set I , a family $(x_i)_{i \in I}$ in E and for some $U \in U(E)$ and some $k \in [1, +\infty]$ the following extended real numbers are considered.

$$N_{U,k}((x_i)_{i \in I}) := \begin{cases} (\sum_I p_U(x_i)^k)^{1/k} & \text{if } k \in [1, +\infty[\\ \sup_I p_U(x_i) & \text{if } k = +\infty, \end{cases}$$

$$M_{U,k}((x_i)_{i \in I}) := \begin{cases} \sup_{x' \in U^\circ} (\sum_I |(x', x_i)|^k)^{1/k} & \text{if } k \in [1, +\infty[\\ \sup_{x' \in U^\circ} \sup_I |(x', x_i)| & \text{if } k = +\infty. \end{cases}$$

where (x', x_i) represents the value of the continuous linear functional x' at x_i . To shorten the notations these two numbers will be written as $N_{U,k}(x_i)$ and $M_{U,k}(x_i)$, respectively.

Given two locally convex spaces E and F the following numbers corresponding to an element $z = \sum_{i=1}^n x_i \otimes y_i$, with $x_i \in E$ and $y_i \in F$ define seminorms on $E \hat{\otimes} F$.

$$\epsilon_{U,V}(z) := \sup \left\{ \left| \sum_{i=1}^n (x', x_i)(y', y_i) \right| ; x' \in U^\circ, y' \in V^\circ \right\}$$

$$\lambda_{U,V,k}(z) := \inf N_{U,k}(x_i) M_{V,k}(y_i)$$

$$\rho_{U,V,k}(z) := \inf M_{U,k}(x_i) N_{V,k}(y_i)$$

$$\pi_{U,V,k}(z) := \inf \sum_{i=1}^n p_U(x_i) p_V(y_i)$$

with $U \in U(E)$ and $V \in U(F)$, where the supremum and the infima are taken over all representations of z .

The seminorms $\lambda_{U,V,k}$ and $\rho_{U,V,k}$ are decreasing functions of $k \in [1, +\infty]$, $\varepsilon_{U,V} \leq \lambda_{U,V,k}$, $\varepsilon_{U,V} \leq \rho_{U,V,k}$ and $\lambda_{U,V,1} = \rho_{U,V,1} = \pi_{U,V}$.

If τ_ε , τ_{λ_k} , τ_{ρ_k} and τ_π denote the locally convex topologies generated by the systems of seminorms $(\varepsilon_{U,V})$, $(\lambda_{U,V,k})$, $(\rho_{U,V,k})$, $(\pi_{U,V})$ when U and V run over $U(E)$ and $U(F)$, respectively, and $E_\varepsilon F$, $E_{\lambda_k} F$, $E_{\rho_k} F$, $E_\pi F$ the corresponding locally convex spaces, then for $k_1 < k_2$ in $]1, +\infty[$ the following inequalities of topologies hold

$$\tau_\varepsilon < \tau_{\lambda_\infty} < \tau_{\lambda_{k_2}} < \tau_{\lambda_{k_1}} < \tau_{\lambda_1} = \tau_\pi,$$

$$\tau_\varepsilon < \tau_{\rho_\infty} < \tau_{\rho_{k_2}} < \tau_{\rho_{k_1}} < \tau_{\rho_1} = \tau_\pi.$$

If E or F is a nuclear space, as τ_ε and τ_π coincide [25] then so do the above topologies.

In the sequel τ denotes one of the equivalent locally convex topologies on $E \otimes F$ when E or F is nuclear.

Now we go back to the dual pairs of nuclear spaces (E, E') and (F, F') we have been considering in this paper and before giving the main theorem we prove the following.

PROPOSITION 2.4.10. $\tilde{\chi}^2(D, F, E') \cong \tilde{\chi}^2(D, F) \hat{\otimes}_\tau E'$ and $L_{F'}^2(\Omega, A, \mathcal{P}) \cong L_R^2(\Omega, A, \mathcal{P}) \hat{\otimes}_\tau F'$.

Proof : Let U represent the unit ball of $\tilde{\chi}^2(D, F)$ and V be a neighborhood in $u_h(E')$. The seminorms $\rho_{U,V,2}$ and $\lambda_{U,V,2}$ on $\tilde{\chi}^2(D, F) \hat{\otimes}_\tau E'$ are equivalent to the Hilbert-Schmidt norm of $\tilde{\chi}^2(D, F) \hat{\otimes}_2 E'(V)$, [26]. If $A = \sum_{k=0}^n x_k \otimes S_k$ is an element of

$\tilde{\Lambda}^2(D, F) \hat{\otimes} E'$, then we define the mapping i from $\tilde{\Lambda}^2(D, F) \hat{\otimes} E'$ into $\tilde{\Lambda}^2(D, F, E')$ by

$$i(A)_t(\omega) := \tilde{A}_t(\omega) = \sum_{k=0}^n (X_{k,t}(\omega), \cdot)_{H_t(\omega)} \tilde{S}_k. \quad \text{According to Proposition 1.2.2, the Hilbert-}$$

Schmidt norm of $k(V) \circ A$ is equal to $q_V(\tilde{A})$, (cf. (2.4.1)).

Therefore i extends to an algebraic and topological isomorphism from

$$\tilde{\Lambda}^2(D, F) \hat{\otimes}_{\lambda_2} E' \cong \tilde{\Lambda}^2(D, F) \hat{\otimes}_{\rho_2} E' \cong \tilde{\Lambda}^2(D, F) \hat{\otimes}_\tau E' \quad \text{into} \quad \tilde{\Lambda}^2(D, F, E').$$

Conversely, let us choose $\tilde{A} \in \tilde{\Lambda}^2(D, F, E')$ as defined by $\tilde{A}_t(\omega) := \sum_{n=0}^{\infty} a_n (X_{n,t}(\omega), \cdot)_{H_t(\omega)} S_n$ where (a_n) , (X_n) and (S_n) are as in (2.4.4). \tilde{A} is the limit of $\tilde{A}_{n,t}(\omega) = \sum_{k=0}^n a_k (X_{k,t}(\omega), \cdot)_{H_t(\omega)} S_k$.

The isomorphic inverse image $i^{-1} \tilde{A}_n$ of \tilde{A}_n converges in $\tilde{\Lambda}^2(D, F) \hat{\otimes}_\tau E'$. This proves the isomorphism $\tilde{\Lambda}^2(D, F, E') \cong \tilde{\Lambda}^2(D, F) \hat{\otimes}_\tau E'$.

The second isomorphism announced in the proposition is proved in a similar way. \square

As a consequence, we see that any element A of $\tilde{\Lambda}^2(D, F, E')$ can be represented by

$$(2.4.10) \quad A = \sum_{k=0}^{\infty} a_k X_k \hat{\otimes} S_k \in \tilde{\Lambda}^2(D, F) \hat{\otimes}_\tau E'$$

where $(a_k) \in l^1$, (X_k) is bounded in $\tilde{\Lambda}^2(D, F)$ and (S_k) is equicontinuous in E' .

Moreover, we also can write

$$(2.4.11) \quad A.M = \sum_{k=0}^{\infty} a_k (X_k.M) \hat{\otimes} S_k \in M^2(R) \hat{\otimes}_\tau E'.$$

Let G be a neighborhood in $U_h(E)$ such that $(S_k) \subset G^\circ$. By Proposition 2.4.5. and by (2.4.10) we can write $A = i(G^\circ) \circ A'$, where A' has the representation :

$$(2.4.12) \quad A' = \sum_{k=0}^{\infty} a_k X_k \hat{\otimes} S_k \in \tilde{\Lambda}^2(D, F) \hat{\otimes}_1 E' [G^\circ]$$

Since $\tilde{\Lambda}^2(D, F) \hat{a}_1 E' [G^\circ] \subset \tilde{\Lambda}^2(D, F) \hat{a}_2 E' [G^\circ] \cong \tilde{\Lambda}^2(D, F, E' [G^\circ])$, A' can be given a representation of type (2.3.8), namely

$$(2.4.13) \quad A'_t(\omega) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{m,n,t}(\omega) \tilde{e}_{m,t}(\omega) \hat{a} k_n$$

where (k_n) is a CONS in $E' [G^\circ]$, and A can be written as

$$(2.4.14) \quad A_t(\omega) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{m,n,t}(\omega) (\tilde{e}_{m,t}(\omega), \cdot)_{H_t(\omega)}^{\sim} i(G^\circ) k_n$$

the series converging in $\tilde{\Lambda}^2(D, F, E')$. Consequently, the martingale A.M can be represented by

$$A.M = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (b_{m,n} \cdot M_m) \hat{a} i(G^\circ) k_n,$$

with $M_m = \tilde{e}_m \cdot M$, and the series converges in $M^2(E, E') \cong L_E^2(\Omega, A, P) \cong M^2(R) \hat{a}_t E'$.

3. EXAMPLES AND APPLICATIONS

3.1. NUCLEAR SPACE VALUED BROWNIAN MOTION

The definition of a distribution-valued Brownian motion and the integration with respect to it was given in [9] and [11]. The definition and properties of an F' -valued Brownian motion are quite similar. In order to avoid the classical localization problem for Brownian motions we suppose that t ranges over the finite interval $[0, T]$.

DEFINITION 3.1.1. Let $W = (W_t, t \in [0, T])$ be an element of $M_c^2(F, F')$ and let Q be a continuous symmetric nonnegative linear operator from F into F' . We say that W is an F' -valued Brownian motion with covariance operator Q if

$$\langle W \rangle_t = tQ.$$

Since $W \in M_c^2(F, F')$ there is a neighborhood $G \in U_h(F')$ such that almost all the trajectories of W belong to $F'[G^\circ]$. Considered as an $F'[G^\circ]$ -valued martingale, one can prove that W is an $F'[G^\circ]$ -valued Brownian motion. As previously, we identify $F(G)$ and $F'[G^\circ]$ and denote them by H . If Q^G is the covariance operator of W considered as an H -valued Brownian motion, then we have $Q = i(G^\circ) \circ Q^G \circ k(G)$.

If D^G is an element of $L^2(H, H)$ such that $Q^G = D^G \circ D^{G*}$ we put $D := i(G^\circ) \circ D^G$. Then a scalar product can be defined on F by $(\phi, \psi) = (D^* \phi, D^* \psi)_H$ with $\phi, \psi \in F$. \tilde{H} is the completion of F in the Hilbertian topology induced by this new scalar product. It is identified with a subspace of the algebraic dual F'^* of F' . Everything we did in the preceding paragraphs for a general square integrable martingale M can be repeated for W by replacing all $\tilde{H}_t(\omega)$ by \tilde{H} , $Q_t d\tilde{M}_t$ by $Q dt$ and $\lambda(dt, d\omega)$ by $dt \mathbb{P}(d\omega)$.

As we have seen in Paragraph 1.3, $\tilde{\Lambda}^2(D, F)$ coincides with $L_H^2(\Omega, \mathcal{P}, \lambda)$. We also remarked that if X is an element of $L_H^2([0, T], \mathcal{B}_{[0, T]}, dt)$, then $X \rightarrow \int_0^T X_t dW_t$ is an isometry of $L_H^2([0, T], \mathcal{B}_{[0, T]}, dt)$ into $L_R^2(\Omega, \mathcal{A}, \mathbb{P})$. Considered as a linear process from $L_H^2([0, T], \mathcal{B}_{[0, T]}, dt)$ into $L^2(\Omega, \mathcal{A}, \mathbb{P})$, W generates a cylindrical probability on $L_H^2([0, T], \mathcal{B}_{[0, T]}, dt)$, called a Hilbert space valued white noise (cf [1]). We will not deal here with problems concerning the cylindrical theory of the white noise. We shall consider the white noise rather as a distribution valued process.

3.2. WHITE NOISE PROCESS

A white noise is also defined as the derivative of the Brownian motion in the sense of distributions. We shall work here on this definition.

Let us consider the dual pair $(\mathcal{D}, \mathcal{D}')$ where \mathcal{D} is the space of infinitely differentiable real functions with compact supports and \mathcal{D}' the space of all distributions on \mathbb{R} both taken in their usual topologies. The pair $(\mathcal{D}, \mathcal{D}')$ has all the properties of the pair (F, F') .

We consider a real valued Brownian motion W on \mathbb{R}_+ . W is a random distribution defined by $W(\phi) := \int_{\mathbb{R}_+} \phi_t W_t dt$, for $\phi \in \mathcal{D}$. Its derivative \dot{W} in \mathcal{D}' is defined by $\dot{W}(\phi) := -W(\dot{\phi}) = \int_{\mathbb{R}_+} \phi_t dW_t$. In this sense \dot{W} is a \mathcal{D}' -valued random variable. But for many applications we need a definition of the white noise rather as a process. We adopt the following one here.

DEFINITION 3.2.1. We call a (real-valued) white noise process the \mathcal{D}' -valued Gaussian process $\dot{W} = (\dot{W}_t, t \in \mathbb{R}_+)$ defined by

$$\phi \rightarrow \dot{W}_t(\phi) = \int_0^t \phi_s dW_s, \quad \phi \in \mathcal{D}.$$

We have $\dot{W} \in M_C^2(\mathcal{D}, \mathcal{D}')$ and $(\langle \dot{W} \rangle_t \phi, \psi) = \langle \phi \cdot W, \psi \cdot W \rangle_t = \int_0^t \phi_s \psi_s ds$ for $\phi, \psi \in \mathcal{D}$. Therefore, we can write $Q_t = \delta_t \otimes \delta_t$ where δ_t is the Dirac distribution at the point t . We already have the factorization of Q_t through the Hilbert space \mathbb{R} . In fact, $D_t: a \in \mathbb{R} \rightarrow a\delta_t \in \mathcal{D}'$ and $D_t^*: \phi \in \mathcal{D} \rightarrow \delta_t(\phi) = \phi(t) \in \mathbb{R}$. We can define a scalar product on \mathcal{D} by

$$(\phi, \psi)_{\tilde{H}_t} := (Q_t \phi, \psi)_{\mathcal{D}', \mathcal{D}} = (\delta_t \phi, \delta_t \psi)_{\mathbb{R}} = \phi_t \psi_t$$

for $\phi, \psi \in \mathcal{D}$. Let us complete \mathcal{D} by the set of all the real functions on \mathbb{R} . Two functions f and g belong to the same equivalence class iff $f_t = g_t$. Divided by this equivalence relation the set of all real functions becomes a Hilbert space denoted by \tilde{H}_t and equipped with the scalar product $(f, g)_{\tilde{H}_t} = f_t g_t$.

Obviously \tilde{H}_t is isometric to \mathbb{R} . Therefore $\tilde{\Lambda}^2(D, \mathcal{D})$ is isometric to $L^2_{\mathbb{R}}(\Omega', \mathcal{P}, \lambda)$ where $\lambda(dt, d\omega) = dt \mathbb{P}(d\omega)$.

Let X be a \mathcal{D} -valued elementary process given by

$$X_t(s, \omega) = \sum_{i=0}^{n-1} 1_{]t_i, t_{i+1}]} \times B_i(t, \omega) \phi_i(s)$$

where $t_0 < t_1 < \dots < t_n$, $B_i \in F_{t_i}$ and $\phi_i \in \mathcal{D}$. According to the definition of the stochastic integral, we put

$$\begin{aligned} \int_{\mathbb{R}_+} X_t d\dot{W}_t &= \sum_{i=0}^{n-1} 1_{B_i} [\dot{W}_{t_{i+1}}(\phi_i) - \dot{W}_{t_i}(\phi_i)] \\ &= \sum_{i=0}^{n-1} 1_{B_i} \int_{t_i}^{t_{i+1}} \phi_i(t) dW_t \\ &= \int_{\mathbb{R}_+} \left[\sum_{i=0}^{n-1} 1_{]t_i, t_{i+1}]} \times B_i(t, \cdot) \phi_i(t) \right] dW_t \\ &= \int_{\mathbb{R}_+} X_t(t, \cdot) dW_t. \end{aligned}$$

We have $E \left[\int_{\mathbb{R}_+} X_t d\dot{W}_t \right]^2 = \int_{\mathbb{R}_+} E[X_t^2(t, \cdot)] dt$. We easily see that the extension of the stochastic integral to the elements of $\tilde{\Lambda}^2(D, \mathcal{D})$ gives, by isometry, the stochastic integrals, in the sense of Ito, of elements of $L^2_{\mathbb{R}}(\Omega', \mathcal{P}, \lambda)$.

We conclude this observation by the following statement.

PROPOSITION 3.2.2. Let $\dot{W} = (\dot{W}_t, t \in \mathbb{R}_+)$ be a real-valued white noise and let I be the isometry of $\tilde{\Lambda}^2(D, \mathcal{D})$ onto $L^2_{\mathbb{R}}(\Omega', \mathcal{P}, \lambda)$ defined above. Then for $X \in \tilde{\Lambda}^2(D, \mathcal{D})$ we have

$$\int_{\mathbb{R}_+} X_t d\dot{W}_t = \int_{\mathbb{R}_+} (IX)_t dW_t.$$

3.3. GIRSANOV THEOREM

We go on using the notations of Section 2.

In order to simplify the expressions, we take $M \in M_C^2(F, F')$. Let us choose an element ℓ of $\tilde{\Lambda}^2(D, F)$ and put

$$Z_t = \exp \left(- \int_0^t \ell_s dM_s - \frac{1}{2} \int_0^t \|\ell_s(\cdot)\|_{H_s(\cdot)}^2 d\langle M \rangle_s \right)$$

If $E(Z_\infty) = 1$, then $Q = Z_\infty \cdot P$ is a probability on \mathcal{A} , equivalent to P . Let h be defined by $h_t(\omega) := D_t(\omega) \cdot I_t(\omega) \ell_t(\omega)$ λ -a.e. (cf. Proposition 2.4.8) and the F' -valued process Y by

$$Y_t := \int_0^t h_s d\langle M \rangle_s + M_t$$

Then under Q , $Y \in M_C^2(F, F')$ and $\langle Y \rangle = \langle M \rangle$.

Proof. The theorem is an immediate consequence of the Girsanov Theorem for Hilbert space valued square integrable martingales, [19]. As in the finite dimensional case, $M + \langle \ell \cdot M, M \rangle = Y$ is a square integrable martingale under Q and $\langle Y \rangle = \langle M \rangle$. But by using the representation (2.4.9) we obtain

$$\langle \ell \cdot M, M \rangle_t = \langle \ell \cdot M, (D \cdot I)M \rangle_t = \int_0^t D_s(\cdot) I_s(\cdot) \ell_s(\cdot) d\langle M \rangle_s = \int_0^t h_s d\langle M \rangle_s. \text{ From this the}$$

conclusion follows. \square

To conclude this short section we would like to mention that the results of [12] can be extended for the derivation of the filtering equation when the process Y given above is the observation process in a filtering scheme.

The complete extension of the results of [17] in the Hilbertian case to general nuclear space models needs the characterization of special

semi-martingales and the definition of their local characteristics. But the theory of nuclear space valued processes has only partial results on this subject, (cf. [30]).

ACKNOWLEDGEMENTS

This work was prepared during the first author's three months' stay at the Center for Stochastic Processes. Warmest thanks go to Professor G. Kallianpur for his kind invitation and to the entire Department of Statistics for its friendly reception.

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